Solution and error estimates using finite differences (p. 43)

Note: A small part of this material is covered in 660, too.

Notation (a slight change):
\[ Au = -au'' + bu' + cu = f \]
with \( a, b, c \) smooth and \( a(x) > 0, c(x) \geq 0 \) in \( \bar{\Omega} \).

We would like to write down an approximation to this equation that would permit us to solve for values of \( u \) at selected points in \([0,1]\).

Unquiz 2: Suppose \( u \) has 4 continuous derivatives. Prove that the central difference approximations satisfy
\[
    u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2),
\]
\[
    u''(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + O(h^2)
\]
for small values of \( h \).

More formally,
\[
    \left| u''(x) - \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \right| \leq Ch^2|u|_{C^4}
\]
and similarly for \( u'(x) \), where
\[
    |u|_{C^4} = \max_{x \in \Omega} |u'''(x)|
\]

So the finite difference approach is to choose mesh points \( x_j = jh \), where \( h = 1/M \) for some large integer \( M \), and solve for \( u_j \approx u(x_j) \) for \( j = 0, 1, \ldots, M \).

Unquiz 3: Consider the equation
\[ Au = -u'' + bu' + u = f \]
where \( b(x) = x \). Let \( M = 5 \), and write the 4 finite difference equations for \( u \) at \( x = .2, .4, .6, \) and .8.

Properties of the finite difference formulation:
• We obtain a system of linear equations \( AU = g \), where \( g \) is determined by the function \( f \) and the boundary conditions.

• \( A \) is \((M - 1) \times (M - 1)\) and tridiagonal. In the \( j \)th row, the main diagonal element is \( 2a_j/h^2 + c_j \) and the off-diagonal elements are \(- (a_j/h^2 \pm b_j/(2h))\). (The book’s \( A \) is \( h^2 \) times ours.)

• For small enough \( h \), the matrix \( A \) is row diagonally dominant: the main diagonal element is at least as big as the sum of the absolute values of the off-diagonal elements. This ensures that the matrix has no zero eigenvalues and therefore a unique solution \( U \) exists.

Now we need an error estimate, which we obtain from

• a discrete maximum principle.

• a stability estimate.

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A discrete maximum principle

**Lemma 4.1 (p. 44):** Assume \( h \) is small enough that \( a_j \pm \frac{1}{2}hb_j \geq 0 \) and that \( AU \leq 0 \).

• (i) If \( c = 0 \), then

\[
\max_j U_j = \max(U_0, U_M).
\]

• (ii) If \( c \geq 0 \) then

\[
\max_j U_j \leq \max(U_0, U_M, 0).
\]

**Proof of (i):** The \( j \)th equation \((1 \leq j \leq M - 1)\):

\[
2a_jU_j/h^2 - (a_j + hb_j/2)U_{j-1}/h^2 - (a_j - hb_j/2)U_{j+1}/h^2 = g_j \leq 0
\]

so

\[
U_j = \frac{h^2}{2a_j}g_j + \frac{a_j - hb_j/2}{2a_j}U_{j+1} + \frac{a_j + hb_j/2}{2a_j}U_{j-1}
\]

\[
\leq \frac{a_j - hb_j/2}{2a_j}U_{j+1} + \frac{a_j + hb_j/2}{2a_j}U_{j-1}.
\]

Suppose \( U_j \) is the maximum. Then \( U_j = U_{j-1} = U_{j+1} \) because the coefficients on the right add to 1. Continuing this reasoning, we see that \( U \) is constant, so the result holds. Therefore, either \( U \) is constant or the max occurs at an endpoint. 

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A stability estimate

We use the \( \infty \)-norm of the vector \( U \):

\[
\|U\|_\infty = \max_j |U_j|.
\]
Lemma 4.2 (p. 45): If $b = 0$, then
\[ \|U\|_\infty \leq \max(|U_0|, |U_M|) + C\|AU\|_\infty, \]
where $C$ depends on $A$ but not $h$ or $U$.

Proof: Let $w(x) = x - x^2$, $W_j = w(x_j)$, and
\[ \alpha = \min_{x \in \Omega} a(x). \]
Then
\[
(AW)_j = (2a_j + h^2 c_j)W_j/h^2 - a_j W_{j-1}/h^2 - a_j W_{j+1}/h^2
\]
\[
= c_j W_j + \frac{a_j(2x_j - 2x_j^2 - (x_j - h) + (x_j + h)^2)}{h^2}
\]
\[
= c_j W_j + 2a_j
\]
\[ \geq 2\alpha. \]

Now let
\[ V_j^\pm = \pm U_j - (2\alpha)^{-1}\|AU\|_\infty W_j, \]
so that
\[ (AV)_j^\pm = \pm (AU)_j - (2\alpha)^{-1}\|AU\|_\infty (AW)_j \leq 0. \]

Since $W_0 = W_M = 0$, we conclude from Lemma 4.1 that
\[ V_j^\pm = \pm U_j - (2\alpha)^{-1}\|AU\|_\infty W_j \leq \max(|U_0|, |U_M|) \]
and therefore
\[ |U_j| \leq \max(|U_0|, |U_M|) + (2\alpha)^{-1}\|AU\|_\infty |W_j|, \]
and since
\[ \max_j |W_j| = \max_j x_j - x_j^2 = \max_j 1/4 - (x_j - 1/2)^2 = 1/4 \]
the result follows with $C = 1/(8\alpha)$. \[ \]

The error in the finite difference solution

Theorem 4.1 (p. 45): If $b = 0$, then
\[ \max_j |U_j - u(x_j)| \leq C h^2 \|u\|_{C^4}. \]

Proof: Let $e_j = U_j - u(x_j)$. Then by Unquiz 2,
\[ |(Ae)_j| \leq C h^2 \|u\|_{C^4}, \]
so the result follows from Lemma 4.2, noting that $e_0 = e_M = 0$. \[ \]

Summary
The finite difference approximation to our problem leads to a system of linear equations to be solved.

The approximation is \( O(h^2) = O(M^{-2}) \), so the more accuracy we need in the solution, the larger the system.

To get approximations to the solution at points between mesh points, we could use interpolation; see van Loan’s text for details.

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Solution and error estimates using finite elements (p. 51)

Notation:

\[
Au = -(au')' + cu = f \text{ in } \Omega = (0, 1)
\]

with \( u(0) = u(1) = 0 \).

Assumptions:

- \( a(x) \) and \( c(x) \) smooth functions.
- \( a(x) \geq \alpha > 0, c(x) \geq 0 \) in \( \bar{\Omega} \).
- \( f \in L_2(\Omega) \).

Recall the variational formulation

\[
a(u, v) = (f, v), \ v \in H^1_0
\]

where

\[
a(u, v) = \int_{\Omega} (au'v' + cuv)dx
\]

\[
(f, v) = \int_{\Omega} fvdx
\]

As in finite differences, we choose a mesh \( 0 = x_0 < x_1 < \ldots < x_M = 1 \).

\[
h_j = x_j - x_{j-1}, \quad K_j = [x_{j-1}, x_j],
\]

\[
h = \max_j h_j.
\]

But rather than solve for \( u \) at the mesh points, we seek an approximate solution of a particular form:

- continuous,
- satisfying the boundary conditions,
- and piecewise linear in each of the subintervals \( K_j \).
We call the space of such functions $S_h$ and note that it is a subset of $H^1_0$, the space where the solution lives.

A convenient basis

We can construct our solution using any basis for $S_h$, but one basis is particularly convenient: the set of hat functions $\phi_i$, $i = 1, \ldots, M - 1$, where

$$
\phi_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{x_i-x_{i-1}} & x \in [x_{i-1}, x_i] \\
\frac{x-x_{i+1}}{x_i-x_{i+1}} & x \in [x_i, x_{i+1}] \\
0 & \text{otherwise}
\end{cases}
$$

These are designed to satisfy $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ if $i \neq j$.

Any function $v \in S_h$ can be written as

$$
v(x) = \sum_{i=1}^{M-1} v_i \phi_i(x)
$$

where $v_i = v(x_i)$.

The resulting equations

Our original problem: Find $u \in H^1_0$ satisfying

$$
a(u, v) = (f, v)
$$

for all $v \in H^1_0$.

Our new problem: Find $u_h \in S_h$ satisfying

$$
a(u_h, v) = (f, v)
$$

for all $v \in S_h$.

Because the $\phi_i$ form a basis, our new problem becomes: Find

$$
u_h = \sum_{i=1}^{M-1} u_i \phi_i(x)
$$

satisfying

$$
a(u_h, \phi_j) = (f, \phi_j)
$$

for $j = 1, \ldots, M - 1$.

Unquiz 4: Write the resulting system of equations $AU = g$ and compare with the answer to Unquiz 3.

Some properties
• This method of constructing the discrete equations is called Galerkin’s method and is characterized by seeking $u_h$ in some subspace of the space $H^1_0$ that contains the solution, and making the residual $a(u_h, v) - (f, v)$ zero on that subspace.

• $A$ is called the stiffness matrix and $g$ is called the load vector.

• $A$ is symmetric (because $a(\phi_i, \phi_j) = a(\phi_j, \phi_i)$) and $V^T AV = a(v, v) > 0$ when

$$v = \sum_{i=1}^{M-1} v_i \phi_i(x) \neq 0.$$ 

Therefore, the solution exists and is unique.

• $A$ is tridiagonal.

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**Error analysis**

The error analysis of the finite element method proceeds in two steps:

• **Step 1:** Show that for every function $u \in H^1_0$, there is a function $\hat{u}_h \in S_h$ that is close to it.

• **Step 2:** Show that the system of equations yields a solution close to $\hat{u}_h$.

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**Step 1: Approximability**

For any $u \in H^1_0$, let $\hat{u}_h \in S_h$ be defined by

$$I_h u \equiv \hat{u}_h = \sum_{i=1}^{M-1} u(x_i) \phi_i(x).$$

(This is the piecewise linear interpolating function.)

A standard result in approximation theory tells us that over the interval $K_j$ we have

$$\|I_h u - u\|_{K_j} \leq C h^2 |u|_{2, K_j},$$

$$\|(I_h u)' - u'\|_{K_j} \leq C h |u|_{2, K_j}.$$

(The proof follows from Taylor series expansions.)

(Remember notation: $|u|_2 = L_2$ norm of $u''$.)

So

$$\|I_h u - u\| = \left( \sum_{j=1}^{M-1} \|I_h u - u\|_{K_j}^2 \right)^{1/2} \leq \left( \sum_{j=1}^{M-1} C^2 h^4 |u|_{2, K_j}^2 \right)^{1/2} \leq Ch^2 \|u\|_2,$$
and similarly
\[ \| (I_h u)' - u' \| \leq Ch \| u \|_2. \]

**Step 2: \( u_h \) is close to \( I_h u \)**

We use the energy norm
\[ \| v \|_a = a(v, v)^{1/2}. \]

**Theorem 5.1a (p. 54):**
\[ (**) \| u_h - u \|_a = \min_{v \in S_h} \| v - u \|_a \]

**A note:** Let \( e = u - u_h \). We know that \( a(u, v) = (f, v) \) and \( a(u_h, v) = (f, v) \) for all \( v \in S_h \), so
\[ (** *) a(e, v) = 0 \]
for all \( v \in S_h \). This means that the error is orthogonal to \( S_h \), or, in other words, \( u_h \) is the orthogonal projection (with respect to the inner product \( a \)) of \( u \) onto \( S_h \), and therefore (**) holds, as we now prove in detail.

**Proof:** Using (** *), we see that for any \( v \in S_h \),
\[ \| e \|_a^2 = a(e, e) = a(e, u - u_h - v) \equiv a(e, u - \hat{v}) \leq \| e \|_a \| u - \hat{v} \|_a, \]
where \( \hat{v} = v + u_h \in S_h \). Therefore, \( \| e \|_a \leq \| u - \hat{v} \|_a \) for all \( \hat{v} \in S_h \). 

**Theorem 5.1b (p. 54):**
\[ \| u_h' - u' \| \leq Ch \| u \|_2. \]

**Proof:**

Notice that if \( v \in H_0^1 \), then
\[
\| v \|_a^2 = \int_0^1 a(x)(v'(x))^2 + c(x)v(x)^2 \, dx \\
\geq \min_{x \in [0,1]} a(x) \int_0^1 (v'(x))^2 \, dx \\
\geq a \| v' \|^2
\]
and
\[
\| v \|_a^2 = \int_0^1 a(x)(v'(x))^2 + c(x)v(x)^2 \, dx \\
\leq \max_{x \in [0,1]} a(x) \int_0^1 (v'(x))^2 \, dx + \max_{x \in [0,1]} c(x) \int_0^1 (v(x))^2 \, dx \\
\leq C \| v' \|^2 + C \| v \|^2 \\
\leq C \| v' \|^2.
\]
where the last step follows from equation (2.17). Thus, 
\[(\ast) \quad \sqrt{\alpha\|v^\prime\|} \leq \|v\|_a \leq C\|v^\prime\|\]
for $v \in H^1_0$. Now, Theorem 5.1a implies 
\[\|e\|_a \leq \|z - u\|_a\]
for all $z \in S_h$, and since $z - u \in H^1_0$, by (\ast) we conclude that 
\[\|e\|_a \leq \|z - u\|_a \leq C\|z' - u'\|\]
Therefore, 
\[\|e\|_a \leq C\min_{z \in S_h} \|z' - u'\|\]
so, using (\ast) again, 
\[\|e^\prime\| \leq C\|e\|_a \leq C\min_{z \in S_h} \|z' - u'\|\]
Now let $z = I_h u$ and use the interpolation bound 
\[\|(I_h u)' - u'\| \leq Ch\|u\|_2.\]

This is nice, but it gives us a result on the energy norm, not the $L_2$ norm, so we need to work a little more.

**Theorem 5.2 (p. 55):** 
\[\|e\| \leq Ch^2\|u\|_2.\]

**Proof:** We use a duality argument.

**Original problem:** Find $u \in H^1_0$ such that $a(u, \phi) = (f, \phi)$ for all $\phi \in H^1_0$.

**Dual problem:** Find $\phi \in H^1_0$ such that $a(w, \phi) = (w, e)$ for all $w \in H^1_0$.

We proved that 
\[\|\phi\|_1 \leq C\|e\|,\]
but it is also true (see (2.22)) that 
\[\|\phi\|_2 \leq C\|e\|.

Now 
\[
(e, e) = \|e\|^2 \quad \text{(definition)}
= a(e, \phi) \quad \text{(a(w, \phi) = (w, e))}
= a(e, \phi - I_h \phi) \quad \text{(orthogonality)}
\leq \|e\|_a \|(\phi - I_h \phi)\|_a \quad \text{Cauchy-Schwarz}
\leq C\|e^\prime\| \|(\phi - I_h \phi)^\prime\| \quad \text{\((\ast)\)}
\leq Ch\|e^\prime\| \|\phi\|_2 \quad \text{\(\text{approximability}\)}
\leq Ch\|e^\prime\| \|e\|, \quad \text{\(\text{previous equation}\)}
\]

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so \( \|e\| \leq Ch\|e'\| \), and by Theorem 5.1b, this is bounded by \( Ch^2\|u\|_2 \).

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**Higher order approximation**

We derived our finite element equation using the space of piecewise linear functions (i.e., piecewise polynomials of degree 1) with a convenient basis, the basis of hat functions.

We could also use higher order polynomials: quadratics, cubics, etc. The basis we choose consists of our old hat functions plus quadratic or cubic hat functions that vanish at all mesh points.

Picture: p. 57.

Because the approximability properties are better, we get higher order estimates for the error: if we use piecewise polynomials of degree \( r - 1 \), then
\[
\|u - u_h\| \leq Ch^r\|u\|_r, \\
\|u' - u'_h\| \leq Ch^{r-1}\|u\|_r,
\]
when \( u \in H^r \).

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**h-p methods**

**Result:** If we want a better approximation, we have two choices:

- decrease \( h \).
- increase \( r \).

The parameter \( r \) is often called \( p \) in the literature, so the resulting adaptive methods are called h-p methods.

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**Summary**

- We have shown existence, uniqueness, and stability of the solution to our ODE-BVP.
- We have introduced several tools for analysis, including
  - the maximum principle,
  - Green’s functions,
  - approximability,
  - duality,
  - the energy norm,
  - regularity.
• We have defined a finite difference approximation to the ODE-BVP, reducing the problem to solving a linear system of equations.

• We showed existence and uniqueness of the finite difference approximation, as well as an error bound.

• Omitted: We could also have applied shooting methods to solve our ODE-BVP (660).

• We have defined a finite element approximation.

• We showed existence and uniqueness of the finite element approximation, as well as an error bound.