AMSC/CMSC 661 Scientific Computing II

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Solution of Ordinary Differential Equations, Initial Value Problems

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These notes are based on the 2003 textbook
of Stig Larsson and Vidar Thomée.

Ordinary Differential Equations, Initial Value Problems

Ordinary Differential Equations, Initial Value Problems = ODE/IVP

The plan:

- A review of the linear problem and initial conditions
- A few numerical methods

A recurring theme: Stability.

Note: We will assume that any matrix **A** that we use has a complete set of eigenvalues and eigenvectors. Almost all matrices do. In that case, **A** can be factored as in an eigendecomposition as

$$\mathbf{A} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{W}^{-1}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with entries λ_j, the eigenvalues of **A**.

The philosophy:

- Some of this material is covered in 660.
- We'll just do what we need to cover IVPs for PDEs.

Reference: Chapter 7.

A review of the linear problem and initial conditions

A single first order ODE

Problem: Find the function $u(t) : \mathcal{R} \to \mathcal{R}$ that satisfies

$$u' + au = f(t)$$

for $t > 0$, with $u(0) = v$, a given number and a a given number.

Jargon: The equation is called first order because the highest derivative is the first.

Solution:

$$u(t) = e^{-at}v + \int_0^t e^{-a(t-s)} f(s) ds.$$

Unquiz 1: Verify that this solution satisfies the differential equation and the initial value condition.

A system of first order ODEs

Problem: Find the function $\mathbf{u}(t) : \mathcal{R}^1 \to \mathcal{R}^n$ that satisfies

$$\mathbf{u}' + \mathbf{A}\mathbf{u} = \mathbf{f}(t)$$

for $t > 0$, with $\mathbf{u}(0) = \mathbf{v}$, a given vector and $\mathbf{A}$ a given $n \times n$ matrix.

Solution:

$$\mathbf{u}(t) = e^{-t\mathbf{A}}\mathbf{v} + \int_0^t e^{-(t-s)\mathbf{A}} \mathbf{f}(s) ds,$$

where

$$e^{\mathbf{B}} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{B}^j.$$

Unquiz 2: Verify that this solution satisfies the differential equation and the initial value condition.

A more useful expression for the matrix exponential function

If $\mathbf{A} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix with entries λ_j, then

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} (\mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1})^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{W}\mathbf{\Lambda}^j\mathbf{W}^{-1} \\ &= \mathbf{W} \left(\sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{\Lambda}^j \right) \mathbf{W}^{-1} \end{aligned}$$

Aside: Taylor series tells us that

$$\begin{aligned} e^{\lambda} &= e^0 + \lambda e^0 + \frac{1}{2}\lambda^2 e^0 + \frac{1}{3!}\lambda^3 e^0 + \dots \\ &= 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \dots \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \lambda^j. \end{aligned}$$

So we conclude that

$$e^{\mathbf{A}} = \mathbf{W} \exp(\mathbf{\Lambda}) \mathbf{W}^{-1}$$

where $\exp(\mathbf{\Lambda})$ is a diagonal matrix with entries e^{λ_j}.

Observation:

- Since the solution to the ODE is

$$\mathbf{u}(t) = e^{-t\mathbf{A}}\mathbf{v} + \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{f}(s)ds,$$

  we see that as $t \to \infty$, $\mathbf{u}(t) \to \mathbf{0}$ if all eigenvalues of $\mathbf{A}$ are positive. This is called asymptotic stability. In this case, small changes in the data make small changes in the solution.

- If $\mathbf{A}$ has any negative eigenvalues, then the solution $\mathbf{u}$ can grow as $t \to \infty$. This is called instability.

A single second order ODE

Problem: Find the function $u(t) : \mathcal{R} \to \mathcal{R}$ that satisfies

$$u'' + au = f(t)$$

for $t > 0$, with $u(0) = v, u'(0) = w$, (given numbers) and a a given number.

Solution:

$$u(t) = \cos(t\sqrt{a})v + \frac{1}{\sqrt{a}}\sin(t\sqrt{a})w$$

if $f = 0$.

Unquiz 3: Verify that this solution satisfies the differential equation and the initial value conditions.

A system of second order ODEs

Problem: Find the function $\mathbf{u}(t) : \mathcal{R}^1 \to \mathcal{R}^n$ that satisfies

$$\mathbf{u}'' + \mathbf{A}\mathbf{u} = \mathbf{f}(t)$$

for $t > 0$, with $\mathbf{u}(0) = \mathbf{v}, \mathbf{u}'(0) = \mathbf{w}$, (given vectors) and $\mathbf{A}$ a given $n \times n$ matrix.

Solution:

$$\mathbf{u}(t) = \cos(t\sqrt{\mathbf{A}})\mathbf{v} + (\sqrt{\mathbf{A}})^{-1}\sin(t\sqrt{\mathbf{A}})\mathbf{w}$$

if $\mathbf{f} = \mathbf{0}$, where

$$\cos \mathbf{B} = \frac{1}{2}(e^{i\mathbf{B}} + e^{-i\mathbf{B}})$$
$$\sin \mathbf{B} = \frac{1}{2i}(e^{i\mathbf{B}} - e^{-i\mathbf{B}})$$
$$\sqrt{\mathbf{A}} = \mathbf{W}\sqrt{\mathbf{\Lambda}}\mathbf{W}^{-1}$$

and $\sqrt{\mathbf{\Lambda}}$ has diagonal entries $\sqrt{\lambda_j}$.

Unquiz 4: Verify that this solution satisfies the differential equation and the initial value conditions.

Three numerical methods for first order equations

- Euler
- backward Euler
- Crank-Nicolson

Numerical methods for a single ODE

The single ODE:

$$u' = f(t, u)$$

Let k be the mesh spacing for t (just as h was for the variable x).

- Euler (an explicit method)

$$\frac{u(t+k) - u(t)}{k} = f(t, u(t)).$$

- backward Euler (an implicit method)

$$\frac{u(t+k) - u(t)}{k} = f(t+k, u(t+k)).$$

- Crank-Nicolson (an implicit method)

$$\frac{u(t+k) - u(t)}{k} = f(t+k/2, (u(t+k) + u(t))/2).$$

(The generalization to a system of equations is clear(???).)

Accuracy of these three methods

- Euler

$$\frac{u(t+k) - u(t)}{k} = f(t, u(t)).$$

Taylor series expansion says we make an error of order k.

– backward Euler

$$\frac{u(t+k)-u(t)}{k} = f(t+k, u(t+k)).$$

Taylor series expansion says we make an error of order k.

– Crank-Nicolson

$$\frac{u(t+k)-u(t)}{k} = f(t+k/2, (u(t+k)+u(t))/2).$$

Taylor series expansion says we make an error of order k^2.

Stability analysis of these three methods

Suppose we apply them to the single ODE

$$u' + au = 0,$$

and let u^n be the approximate solution we obtain for $t = nk$. Then Euler's method gives

$$u(t+k) = u(t) - kau(t)$$

so

$$\begin{aligned} u^0 &= v \\ u^1 &= (1-ka)v \\ u^n &= (1-ka)^n v \end{aligned}$$

Unquiz 5: Show that the Backward Euler method gives

$$u^n = \frac{1}{(1+ka)^n} v. []$$

In a similar way, you could show that Crank-Nicholson gives

$$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v.$$

So what?

Euler:	$u^n = (1-ka)^n v$
Backward Euler:	$u^n = \frac{1}{(1+ka)^n} v$
Crank-Nicholson:	$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v$
True solution:	$u(nk) = e^{-nka} v$

Suppose $a > 0$, so that the ODE is asymptotically stable. When are our approximations asymptotically stable?

Euler:	$u^n = (1 - ka)^n v$ Stable if $\|1 - ka\| < 1$, or $k < 2/a$.
Backward Euler:	$u^n = \frac{1}{(1+ka)^n} v$ Stable unconditionally.
Crank-Nicholson:	$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v$ Stable unconditionally.

A numerical method for a second order equation

$$u'' = f(t, u)$$

becomes

$$\frac{u(t+k) - 2u(t) + u(t-k)}{k^2} = f(t, u(t)).$$

The error is $O(k^2)$, and the method is stable on the linear problem $u'' + au = 0$ for any k and a.