AMSC/CMSC 661 Quiz 1 , Spring 2010

1. (10) Consider the differential equation

$$\mathcal{A}u \equiv -(au')' + bu' + cu = f \text{ in } \Omega = (0,1)$$

$$u(0) = 0, \;\; u(1) = 0,$$

where the functions $a(x) > 2, c(x) \geq 0, b(x)$, and $f(x)$ are smooth on the interval $x \in [0,1]$. How sensitive is the solution to small changes in the boundary conditions? Specifically, if we change the problem to $\mathcal{A}v = f$ with boundary conditions $v(0) = \epsilon_1 > 0$ and $v(1) = \epsilon_2 > 0$, for small numbers ϵ_1, ϵ_2, how much does v differ from u? Justify your answer.

(This situation could arise in modeling a flat plate in thermal equilibrium except that it is difficult to maintain the boundary at a fixed temperature.)

Answer: Let $w(x) = u(x) - v(x)$. Then w satisfies

$$\mathcal{A}w = 0, \;\; w(0) = -\epsilon_1, \;\; w(1) = -\epsilon_2.$$

By Theorem 2.2,

$$\|w\|_C \leq \max(|w(0)|, |w(1)|) + C\|\mathcal{A}w\|_C = \max(\epsilon_1, \epsilon_2).$$

Therefore,

$$\max_{x\in[0,1]} |u(x) - v(x)| = \max(\epsilon_1, \epsilon_2).$$

2. (10 points) consider the probleM

$$-u''(x) - u'(x) = e^{1-x},$$

$$u(0) = 0, \;\; u(1) = 1.$$

Use the maximum principle and the minimum principle, to tell me as much as you can about the solution to this problem, without actually solving the problem.

Answer: In the book's notation, $a(x) = 1 > 0$, $b(x) = -1$, $c(x) = 0$, $f(x) = e^{1-x} > 0$ for $x \in \Omega$.

Therefore, the Minimum Principle applies: for $x \in \bar{\Omega}$,

$$u(x) \geq \min(0, 1) = 0.$$

We can't apply the Maximum Principle to this problem, but let $w(x) = -u(x)$. Then

$$-w''(x) - w(x) = -e^{1-x}, \;\; w(0) = 0, \;\; w(1) = -1.$$

Verify the hypotheses and apply the Maximum Principle to obtain

$$w(x) \leq \max(0, -1) = 0,$$

so $u(x) \geq 0$ – no new information.

- We could also use Theorem 2.2, but the bound is not computable.
- Now that we know the monotonicity theorem, we could let $v(x) = x$, so that $-v''(x) - v'(x) = -1$, with $v(0) = 0$, $v(1) = 1$. Applying the monotonicity theorem yields $u(x) \geq x$.
- The true solution is $u(x) = xe^{1-x}$.