AMSC/CMSC 661 Quiz 3 , Spring 2010

1. (10) Let $\Omega = (0,1)$. Give the variational form (weak form) of the problem

$$-u''(x) = f(x), \quad x \in \Omega,$$

$$u(0) = 0, \quad u'(1) = 0.$$

You will need the integration-by-parts formula

$$\int_0^1 v'w\,dx = vw\,|_0^1 - \int_0^1 vw'\,dx.$$

Your final result should look like

$$a(u,\phi) = (f,\phi)$$

for all ϕ in an appropriately chosen space. Tell me what that space is, and how $a(u,\phi)$ is defined.

Answer: Let's take a test function ϕ and integrate its product with the differential equation:

$$\begin{aligned}
(f,\phi) &\equiv \int_0^1 f(x)\phi(x)\,dx \\
&= \int_0^1 (-u''(x))\phi(x)\,dx \\
&= \int_0^1 u'(x)\phi'(x)\,dx - u'(x)\phi(x)\,|_0^1 \\
&= \int_0^1 u'(x)\phi'(x)\,dx \\
&\equiv a(u,\phi).
\end{aligned}$$

To derive the 3rd line, we used integration-by-parts with $v = u'$ and $w = \phi$, and for the 4th line we used the fact that $u'(1) = 0$. Also in that line, we force $\phi(0) = 0$ by choosing ϕ in the space

$$H_\ell^1 = \{v \in H^1(\Omega) : v(0) = 0\}.$$

With these definitions, our variational problem is to find $u \in H_\ell^1$ such that

$$a(u,\phi) = (f,\phi)$$

for all $\phi \in H_\ell^1$.

(This is similar to Problem 2 on 2005 Quiz 2.) Note that if you use the space H_0^1, you have replaced the $u'(1) = 0$ boundary condition by $u(1) = 0$ and solved the wrong problem.

2. Let Ω be a region in $\mathcal{R}^2$ with a smooth boundary Γ. (For example, Γ could be the unit circle and Ω could be its interior.) Actually, I meant that the region should be contained in the unit circle. Sorry for the confusion. I gave an extra point if you made any reasonable assumption about Ω, with a max of 20 points for the quiz. For each of these problems, use either the maximum principle or the minimum principle (verifying the hypotheses) to give a bound on the solution to the differential equation.

2a. (5)

$$u_{xx} + u_{yy} = 1 - x^2 - y^2, \;\; (x, y) \in \Omega,$$
$$u(x, y) = 0, \;\; (x, y) \in \Gamma.$$

Answer: We need to rewrite this as

$$-u_{xx} - u_{yy} = -1 + x^2 + y^2, \;\; (x, y) \in \Omega,$$

Then

- $a(x, y) = 1 > 0$ and $c(x, y) = 0$, so the assumptions global for the chapter are satisfied.
- $Au = f(x, y) = -1 + x^2 + y^2 \leq 0$ inside the unit circle.

Therefore, by the Maximum Principle, the maximum of u occurs on the boundary, so

$$u(x, y) \leq 0, \;\; (x, y) \in \Omega.$$

2b. (5)

$$-w_{xx} - w_{yy} + 5w = 1 - x^2 - y^2, \;\; (x, y) \in \Omega,$$
$$w(x, y) = 5, \;\; (x, y) \in \Gamma.$$

Answer:

- $a(x, y) = 1 > 0$ and $c(x, y) = 5 > 0$, so the assumptions global for the chapter are satisfied.
- $Aw = f(x, y) = 1 - x^2 - y^2 \geq 0$ inside the unit circle.

Therefore, by the Minimum Principle,

$$w(x, y) \geq \min(\min_{(x,y)\in\Gamma} w(x, y), 0) = 0 \;\; (x, y) \in \Omega.$$