1. (6) The following is the CG algorithm for solving the linear system $Ax = b$. As usual, $A$ is a symmetric positive definite matrix with $nz$ nonzero elements and $b$ is a vector of length $n$. How much storage does the algorithm use? How many multiplications and divisions does it perform? (Express your answers in terms of the parameters $n$, $K$, and $nz$.)

Let $r = b$, $x = 0$, and let $\gamma = r^T r$, $p = r$.
for $k = 1, \ldots, K$

\[
q = Ap
\]
\[\alpha = \gamma/(p^T q)\]
\[x = x + \alpha p\]
\[r = r - \alpha q\]
\[\hat{\gamma} = r^T r\]
\[\beta = \hat{\gamma}/\gamma, \gamma = \hat{\gamma}\]
\[p = r + \beta p\]

end (for $k$)

Answer: We need storage for matrix $A$ and vectors $x, r, b, p, q$, for a total of $nz+5n$ floating-point numbers plus a few scalars, and (in MATLAB) $2nz$ integers.

In each iteration, we need one matrix-vector product, 2 inner products (for $\alpha$ and $\hat{\gamma}$), and 3 vector updates ($x, r, p$) for a total of $K(nz + 5n) + n$ multiplications plus $2K$ divisions.
2a. (6) Suppose we use CG to solve the linear system \( \hat{G}x^* = c \), and we know that all of the eigenvalues of \( \hat{G} \) are in the interval \([1, 10^4]\). Give an upper bound on the number of iterations it will take to reduce the error
\[
\|x^{(k)} - x^*\|_G
\]
by a factor of \(10^5\).

**Answer:** Let \( e_k = \|x^{(k)} - x^*\|_G \). Let \( \kappa \) be the ratio of the largest eigenvalue of \( \hat{G} \) to the smallest. The convergence theorem for CG says
\[
e_k \leq 2 \left( \frac{\kappa - 1}{\kappa + 1} \right)^k e_0.
\]
We know that \( \kappa \leq 10^4 \). Therefore,
\[
e_k \leq 2 \left( \frac{100 - 1}{100 + 1} \right)^k e_0.
\]
We need
\[
2 \left( \frac{100 - 1}{100 + 1} \right)^k \leq 10^5,
\]
or
\[
\left( \frac{100 - 1}{100 + 1} \right)^k \leq \frac{10^5}{2}.
\]
Taking logs of both sides (base \( e \) or base 10 would be fine for a calculator) gives
\[
k(\log 99 - \log 101) \leq \log(10^5/2).
\]
Since \( k \) must be an integer, we set
\[
k \geq \left\lceil -\frac{\log(10^5/2)}{\log 99 - \log 101} \right\rceil,
\]
and, with a calculator we could find that we need at most \( k = 541 \) iterations. (This only makes sense if \( n \geq 541 \); otherwise, without rounding error, the iteration will stop after \( n \) iterations with the exact solution.)
2b. (8) Apply one step of SOR \((\omega = 1.5)\) to the linear system

\[
\begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1^* \\
x_2^*
\end{bmatrix}
= \begin{bmatrix}
4 \\
1
\end{bmatrix}
\]

with a starting guess of \(x_1^{(0)} = x_2^{(0)} = 1\). What is \(x^{(1)}\)? Will the iteration converge to the true solution? Justify your answer.

Answer: For Gauss-Seidel,

\[
\begin{align*}
x_1^{(1)\, GS} &= (4 + 1)/2 = 2.5 \\
x_2^{(1)\, GS} &= (1 + 2.5)/2 = 1.75
\end{align*}
\]

Therefore, the SOR iterate is

\[
x^{(1)} = (1 - \omega)x^{(0)} + \omega x^{(1)\, GS} = -.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1.5 \begin{bmatrix} 2.50 \\ 1.75 \end{bmatrix} = \begin{bmatrix} 3.250 \\ 2.125 \end{bmatrix}.
\]

The eigenvalues of \(A\) are roots of

\[
\det(A - \lambda I) = 0.
\]

Therefore,

\[
(2 - \lambda)^2 - 1 = 0
\]

or

\[
2 - \lambda = \pm 1,
\]

so

\[
\lambda = 1, 3.
\]

Therefore matrix \(A\) is symmetric and positive definite. (Gershgorin’s theorem would also tell us this.)

The “Convergence of SIMs” handout (http://www.cs.umd.edu/users/oleary/c661/simnotes.pdf) says that SOR is convergent for this \(\omega\) when \(A\) is symmetric and positive definite.