AMSC/CMSC 661 Quiz 8 , Spring 2010

1. (10) Let $\Omega = \{\boldsymbol{x} : -2 \leq x_1 \leq 1, \;\; -1 \leq x_2 \leq 1\}$, and let $\Gamma(\Omega)$ be the boundary of Ω. Let

$$q(\boldsymbol{x}, t) = 3x_1 - 2t(x_2 - 1)(x_1 + 1) - 6t + 2t^2.$$

Consider the problem

$$\begin{aligned}
\frac{\partial u(\boldsymbol{x}, t)}{\partial t} - \Delta u(\boldsymbol{x}, t) = &\quad q(\boldsymbol{x}, t) &\quad \text{for } \boldsymbol{x} \in \Omega \subset \mathcal{R}^2, t \in \mathcal{R}_+ \\
u(\boldsymbol{x}, 0) = &\quad 3x_1 &\quad \text{for } \boldsymbol{x} \in \Omega \\
u(\boldsymbol{x}, t) = &\quad t x_1 x_2 &\quad \text{for } \boldsymbol{x} \in \Gamma(\Omega), t \in \mathcal{R}_+
\end{aligned}$$

Give a bound on

$$\max_{0 \leq t \leq 5} \max_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x}, t)|.$$

Answer: We use Theorem 8.7. The dimension of $\boldsymbol{x}$ is $d = 2$. We need three bounds. I obtained the first from the triangle inequality.[1]

$$\begin{aligned}
\max_{\substack{0 \leq t \leq 5 \\ \boldsymbol{x} \in \Omega}} |q(\boldsymbol{x}, t)| &\leq 6 + 10 * 2 * 2 + 30 + 50 = 126, \\
\max_{\boldsymbol{x} \in \Omega} |u(x, 0)| &= 6, \\
\max_{\substack{0 < t \leq 5 \\ \boldsymbol{x} \in \Gamma(\Omega)}} |t x_1 x_2| &= 5 * 2 = 10.
\end{aligned}$$

We need the radius of a circle that contains Ω. If we put the center of the circle at the center of the rectangle, its diameter is $\sqrt{3^2 + 2^2} = \sqrt{13}$.

Therefore,

$$\begin{aligned}
\|u\|_{\mathcal{C}(\bar{\Omega} \times [0,T])} &\leq \max(\|g\|_{\mathcal{C}(\Gamma \times [0,T])}, \|v\|_{\mathcal{C}(\bar{\Omega})}) + \frac{r^2}{2d} \|f\|_{\mathcal{C}(\bar{\Omega} \times [0,T])} \\
&\leq 10 + \frac{13}{4 * 4} * 126 \leq 113.
\end{aligned}$$

[1] More careful analysis shows that the max occurs at $x_1 = 1$, $x_2 = -1$, and $t = 5$, giving the bound $|q(\boldsymbol{x}, t)| \leq 63$.

2. (10) Suppose that the problem

$$\begin{aligned} -\Delta w &= f(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega, \\ w(\boldsymbol{x}) &= 0 && \text{for } \boldsymbol{x} \in \Gamma(\Omega) \end{aligned}$$

has been discretized using piecewise linear finite elements (with basis functions ϕ_j) to obtain the linear system $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{f}$. The $2 \times n$ matrix $\boldsymbol{P}$ contains the coordinates of the vertices of the triangles of the mesh, the vector $\boldsymbol{w}$ contains the values of the approximate solution at the vertices, and $f_j = (f, \phi_j)$. Also suppose that you are given a matrix $\boldsymbol{B}$, the mass matrix, with entries $b_{ij} = (\phi_i, \phi_j)$.

Now suppose we want to solve

$$\begin{aligned} u_t - \Delta u &= f(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega, t \geq 0, \\ u(\boldsymbol{x}, 0) &= v(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x}, t) &= 0 && \text{for } \boldsymbol{x} \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{aligned}$$

Write MATLAB code to obtain an approximate solution to this problem for $t = 0, 0.1, 0.2, \ldots 1.0$ using the backward Euler method in time and finite elements in space. (If this is too confusing, you can get up to 7 points credit for forward Euler.)

Answer: The backward Euler method is

$$\boldsymbol{B}\left(\frac{\boldsymbol{u}^{m+1} - \boldsymbol{u}^m}{k}\right) + \boldsymbol{A}\boldsymbol{u}^{m+1} = \boldsymbol{f}.$$

Collecting terms we get

$$\left(\frac{1}{k}\boldsymbol{B} + \boldsymbol{A}\right)\boldsymbol{u}^{m+1} = \boldsymbol{f} + \frac{1}{k}\boldsymbol{B}\boldsymbol{u}^m.$$

Here is the MATLAB code.

```
k = 0.1;
Bk = B / k;
[L,U] = lu(Bk + A);
u(:,1) = v(P)';
for m = 1:10,
    u(:,m+1) = U \ ( L \ ( f + Bk*u(:,m) ) );
end
```

Notice that it is best to factor once (at a cost of $O(n^3)$, where n is the number of meshpoints. Then we can use the LU factors to solve the linear system with forward- and back-substitution at a cost of only $O(n^2)$.