AMSC/CMSC 661 Quiz 9 , Spring 2010

1. (10) Suppose we want to solve the differential equation

$$u_t + (1+x)u_x + 5u = t\cos(x),$$

for $t > 0$, $x \in (0,1)$, with given initial and boundary conditions. Let u_j^n be our approximation to $u(jh, nk)$, where k is the timestep and $h = 1/m$ is the spatial step. Consider the Wendroff Box Scheme finite difference method

$$\begin{aligned}\frac{u_j^{n+1} + u_{j+1}^{n+1} - u_j^n - u_{j+1}^n}{2k} + (1+x_j)\,\frac{u_{j+1}^{n+1} + u_{j+1}^n - u_j^{n+1} - u_j^n}{2h}& \\ +5\,\frac{u_j^{n+1} + u_{j+1}^{n+1} + u_j^n + u_{j+1}^n}{4} &= t_n\cos(x_j).\end{aligned}$$

Given values u_j^n, $j = 0, \ldots, m$, explain how you would compute u_j^{n+1}, $j = 0, \ldots, m$.

Answer: We would need to be given the boundary condition

$$u_0^{n+1} = u(0, (n+1)k).$$

This is the relevant boundary condition, since the coefficient of u_x is positive, so the characteristics slope upward to the right. We solve the equation for u_{j+1}^{n+1}:

$$\begin{aligned}\left(\frac{1}{2k} + \frac{1+x_j}{2h} + \frac{5}{4}\right) u_{j+1}^{n+1} &= t_n\cos(x_j) - \frac{u_j^{n+1} - u_j^n - u_{j+1}^n}{2k} \\ &\quad -(1+x_j)\,\frac{u_{j+1}^n - u_j^{n+1} - u_j^n}{2h} \\ &\quad -5\,\frac{u_j^{n+1} + u_j^n + u_{j+1}^n}{4}\end{aligned}$$

We divide through by the coefficient of u_{j+1}^{n+1}. This gives us an equation we can evaluate for $j = 0, 1, ..., m-1$, for a given value of n.

2. Consider the problem

$$\begin{aligned} u_t + (1+t)u_x &= 0, \quad x \in [0,\infty), \ \ t \in (0,\infty), \\ u &= x^2, \quad \{x \in [0,\infty), \ \ t = 0\} \cup \{x = 0, \ \ t > 0\}. \end{aligned}$$

2a. (5) Write the differential equation that defines the characteristics for this problem.

Answer: Using the coefficients of u_t and u_x on the right-hand sides, we obtain

$$\frac{dt(s)}{ds} = 1,$$

$$\frac{dx(s)}{ds} = 1 + t(s) = 1 + t.$$

2b. (5) Write the solution to the problem.

Answer: This problem is the example on p. 173 of your textbook.