

Control Seminar

March 23, 2004

1 Laplace Transformation

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$f(t)$	$F(s)$
$f(t)$	$\int_0^{\infty} e^{-st} f(t) dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
$f(t-T)\theta(t-T) = \begin{cases} f(t-T) & t \geq T \\ 0, & t < T \end{cases}$	$e^{-Ts} F(s)$
$\frac{d}{dt} f(t)$	$sF(s) - f(0)$
$\frac{d^n}{dt^n} f(t)$	$s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$
1	$\frac{1}{s}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{a^2+s^2}$
e^{-at}	$\frac{1}{s+a}$
$\frac{-e^{at}+e^{bt}}{b-a}$	$\frac{1}{(s-a)(s-b)}$
$\frac{-ae^{at}+be^{bt}}{b-a}$	$\frac{s}{(s-a)(s-b)}$

The general procedure for solving a system of linear differential equation by the Laplace transformation is:

- Transform the system from the time space into the Laplace space. There it becomes a system of algebraic equations
- Solve the system of algebraic equation in the Laplace space
- Re-transform the solution back into the time domain. For the re-transformation from the Laplace into the time domain there exist also an integral transformation:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{st} F(s) ds \quad (1)$$

However the direct application of that transformation is quite complicated. Therefore one normally converts $F(s)$ into a sum of terms for which the Laplace transformation can be looked up in a table.

In the context of converting a function $F(s)$ into a sum of terms with known Laplace transformations, the partial fraction expansion of a rational function is probably the most important conversion. The idea of that conversion is, that given a rational function

$$G(s) = \frac{A(s)}{B(s)} \quad (2)$$

where the order of the nominator polynomial $A(s)$ is less than the order n of the denominator polynomial $B(s)$, then one can always write $G(s)$ as a sum of n partial fractions:

$$G(s) = \sum_{i=1}^n \frac{c_i}{s - s_i} \quad (3)$$

where the s_i are the n zero points of the denominator polynomial $B(s)$, which are also called poles of the rational function $G(s)$. The calculation of the c_i is given by

$$c_i = \lim_{s \rightarrow s_i} (G(s)(s - s_i)) \quad (4)$$

This formula can be easily derived by multiplying (3) with $(s - s_i)$ and evaluating then both sides at s_i .

Remark 1.1 *In case the order of the nominator polynomial is equal to the denominator one has to make a polynomial division before doing the partial fraction expansion.*

2 Example

2.1 Issue

Calculate the general solution for the system:

$$\ddot{x} + 2d\omega_0\dot{x} + \omega_0^2x = 0 \quad (5)$$

with the initial conditions:

$$x(0) = x_0 \quad \dot{x}(0) = v_0 \quad (6)$$

Use afterwards the values:

case	d	ω_0	x_0	v_0
a)	0	2	1	0
b)	0.5	2	1	1
c)	2	2	1	1

2.2 Solution

At first equation (5) will be transformed to the Laplace space:

$$s^2X - sx(0) - \dot{x}(0) + 2d\omega_0sX - 2d\omega_0x(0) + \omega_0^2X = 0 \quad (7)$$

where $X = X(s)$ denotes the Laplace transform of $x = x(t)$. Using now the initial conditions (6) we get:

$$X(s^2 + 2d\omega_0s + \omega_0^2) = sx_0 + v_0 + 2d\omega_0x_0 \quad (8)$$

$$X = \frac{sx_0}{s^2 + 2d\omega_0s + \omega_0^2} + \frac{v_0 + 2d\omega_0x_0}{s^2 + 2d\omega_0s + \omega_0^2} \quad (9)$$

$$= \frac{sx_0}{(s - s_1)(s - s_2)} + \frac{v_0 + 2d\omega_0x_0}{(s - s_1)(s - s_2)} \quad (10)$$

with s_1 and s_2 being the zero points of the characteristic polynomial $s^2 + 2d\omega_0s + \omega_0^2$:

$$s_{1,2} = \omega_0 \left(-d \pm \sqrt{d^2 - 1} \right) \quad (11)$$

Transforming equation (10) back into the time domain, we have the general solution to (5):

$$x = \frac{-s_1e^{s_1t} + s_2e^{s_2t}}{s_2 - s_1}x_0 + \frac{-e^{s_1t} + e^{s_2t}}{s_2 - s_1}(v_0 + 2d\omega_0x_0) \quad (12)$$

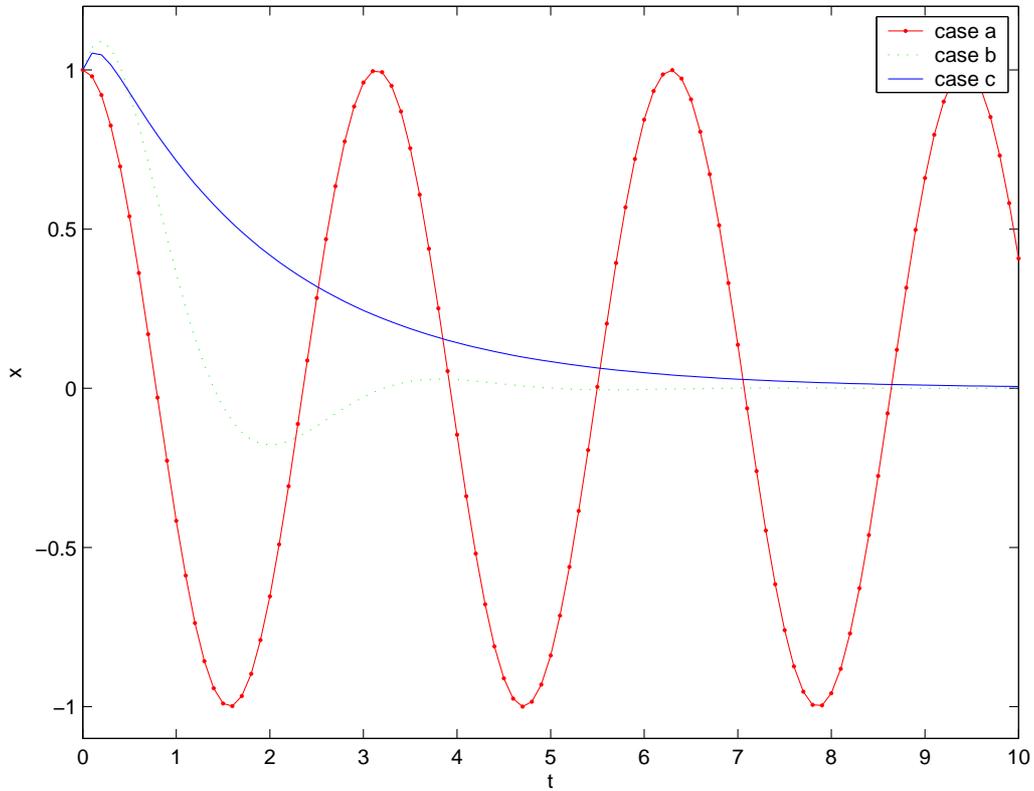


Figure 1: Plots to the example

- a) For the first case one gets $s_1 = 2j, s_2 = -2j$. Hence by using equation (12) the solution in the time domain becomes:

$$\begin{aligned}
 x &= \frac{-2j(\cos(2t) + j \sin(2t)) - 2j(\cos(-2t) + j \sin(-2t))}{-4j} \cdot 1 \\
 &\quad + \frac{-(\cos(2t) + j \sin(2t)) + (\cos(-2t) + j \sin(-2t))}{-4j} (0 + 2 \cdot 0 \cdot 2 \cdot 1) \\
 &= \cos(2t)
 \end{aligned} \tag{13}$$

- b) In the second case we have $s_1 = -1 + j\sqrt{3}, s_2 = -1 - j\sqrt{3}$. Thus we

get:

$$\begin{aligned}
 x &= \frac{(1 - j\sqrt{3})(\cos(\sqrt{3}t) + j\sin(\sqrt{3}t)) + (-1 - j\sqrt{3})(\cos(-\sqrt{3}t) + j\sin(-\sqrt{3}t))}{-2j\sqrt{3}} e^{-t} \cdot 1 \\
 &+ \frac{-\cos(\sqrt{3}t) - j\sin(\sqrt{3}t) + \cos(-\sqrt{3}t) + j\sin(-\sqrt{3}t)}{-2j\sqrt{3}} e^{-t} (1 + 2 \cdot 0.5 \cdot 2 \cdot 1) \\
 &= \frac{-2j\sqrt{3}\cos(\sqrt{3}t) + 2j\sin(\sqrt{3}t)}{-2j\sqrt{3}} e^{-t} + \frac{-2j\sin(\sqrt{3}t)}{-2j\sqrt{3}} 3e^{-t} \\
 &= \left(\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}} \sin(\sqrt{3}t) \right) e^{-t} \tag{14}
 \end{aligned}$$

c) In the last case one has $s_1 = -4 + 2\sqrt{3}$, $s_2 = -4 - 2\sqrt{3}$. This leads to:

$$\begin{aligned}
 x &= \frac{-(-4 + 2\sqrt{3})e^{(-4+2\sqrt{3})t} + (-4 - 2\sqrt{3})e^{(-4-2\sqrt{3})t}}{-4\sqrt{3}} \cdot 1 \\
 &+ \frac{-e^{(-4+2\sqrt{3})t} + e^{(-4-2\sqrt{3})t}}{-4\sqrt{3}} (1 + 2 \cdot 2 \cdot 2 \cdot 1) \\
 &= \frac{5 + 2\sqrt{3}}{4\sqrt{3}} e^{(-4+2\sqrt{3})t} + \frac{-5 + 2\sqrt{3}}{4\sqrt{3}} e^{(-4-2\sqrt{3})t} \tag{15}
 \end{aligned}$$

3 Homework

1. Calculate the Laplace transformations of:

- $\int_0^t f(\tau) d\tau$
- 1
- $\cos(at)$
- $\frac{-e^{at} + e^{bt}}{b-a}$
- $\frac{-ae^{at} + be^{bt}}{b-a}$

2. Calculate the solutions for:

- $\dot{x} - x = 0$ with $x_0 = 1$
- $\dot{x} + x = 0$ with $x_0 = 1$
- $\ddot{x} - 2\dot{x} + 4x = 0$ with $x_0 = 0, v_0 = \dot{x}_0 = 1$
- $\ddot{x} + 2\dot{x} - 4x = 0$ with $x_0 = 0, v_0 = \dot{x}_0 = 1$

3. Implement the four systems above in Simulink

4 Transfer function

For a linear system with a single input and a single output (SISO) the transfer function describes the ratio between the output and the input of the system in the Laplace space. For instance given the system governed by the linear differential equation:

$$\ddot{y} + a_1\dot{y} + a_0y = u \quad (16)$$

which can be written in the Laplace space as:

$$Y(s^2 + a_1s + a_0) = U \quad (17)$$

one gets the transfer function:

$$G(s) = \frac{Y}{U} = \frac{1}{s^2 + a_1s + a_0} \quad (18)$$

Remark 4.1 *For deriving the transfer function of a linear system in the Laplace space all initial states are set to zero!*

Transfer functions are very convenient for describing a system graphically by a block diagram. There they can be combined as shown in Figure 2.

5 Inserting Control

The next goal will be to manipulate the behavior of the system, by applying a controller to it. A normal configuration for a controlled system is shown in Figure 3. The input-output relation of such a system can be described in the Laplace space by:

$$y = G(s)u \quad (19)$$

$$u = C(s)v = C(s)(r - y) \quad (20)$$

$$\Rightarrow y = \frac{C(s)G(s)}{1 + C(s)G(s)}r \quad (21)$$

Equation (19) is normally referred to as the transfer function of the (uncontrolled) plant, while equation (21) represents the transfer function of the controlled plant or the *closed loop* system respectively. The control law for the system is given by equation (20).

In the following we want to study more detailed how to chose the transfer function $C(s)$ in the control law (20) in order to manipulate the closed loop system behavior. The most common controllers are build up by summing

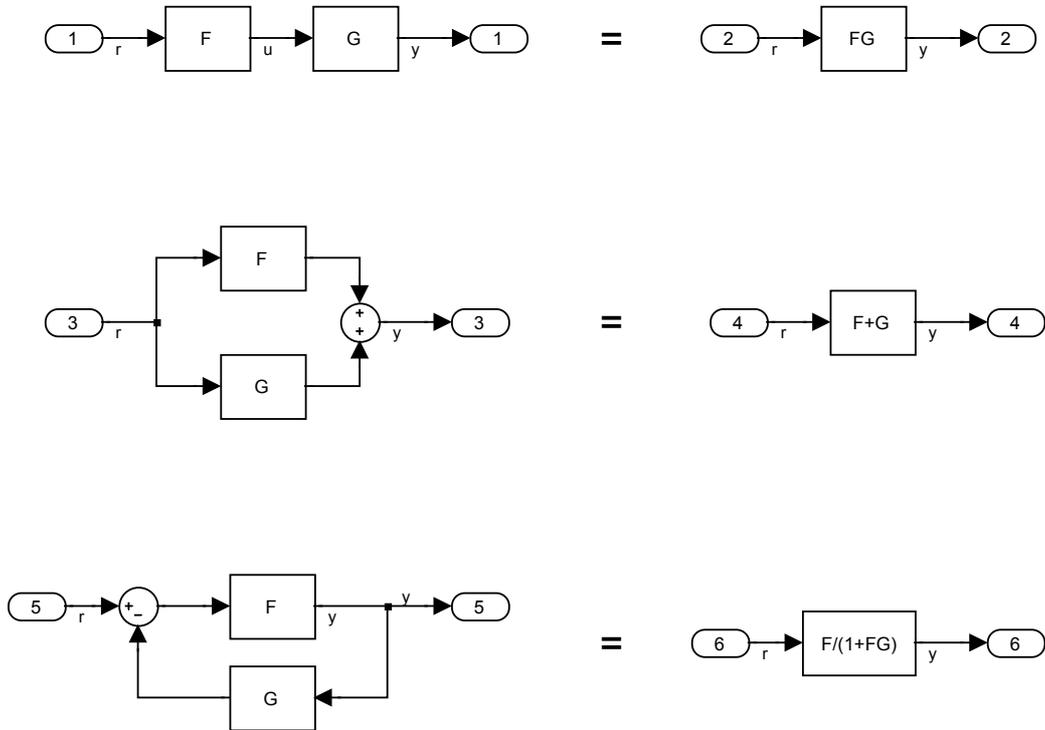


Figure 2: Combining Transfer functions

the three elementary control units:

Name	Transfer function	equation in the time domain
P (proportional)	K_P	$u = K_P v$
I (integral)	$\frac{1}{s} K_I$	$u = K_I \int_0^t v dt$
D (derivative)	$s K_D$	$u = K_D \frac{d}{dt} v$

A common criteria for evaluating the performance of a controller is the response of the closed loop system upon a step input:

$$r(t) = \theta(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \quad (22)$$

By regarding the step response of the closed loop system, the manipulation capability of different controllers shall be demonstrated. To begin with a P

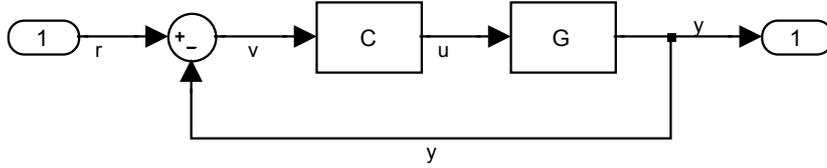


Figure 3: Controlled system

controller shall be applied to the second order plant:

$$\ddot{y} + a_1\dot{y} + a_0y = u \quad \rightarrow \quad G(s) = \frac{y(s)}{u(s)} = \frac{1}{s^2 + a_1s + a_0} \quad (23)$$

Based on equation (21) the transfer function of the closed loop system becomes:

$$\frac{y}{r} = \frac{\frac{K_p}{s^2 + a_1s + a_0}}{1 + \frac{K_p}{s^2 + a_1s + a_0}} = \frac{K_p}{s^2 + a_1s + a_0 + K_p} \quad (24)$$

The poles of the transfer function are determining the denominators of its partial fraction expansion, which can be related directly to the exponents of the exponential functions in the time domain. Therefore the poles of a transfer function provide some significant information about the system behavior. For the transfer function (24) the poles can be expressed analytically as:

$$s_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - (a_0 + K_P)} \quad (25)$$

Due to the importance of the poles for the dynamical behavior of a system, the dependence of their locus upon a parameter of the controller (K_P) is therefore sometimes expressed graphically in the so called *root locus* plot. An example for such a plot is given by Figure 4. The plots for the step response are shown in Figure 5. There the dotted lines indicate the final value of the system output. With regard to that Figure one can observe, that an increase of K_P leads to:

- a faster increase of the system output,
- a smaller steady-state error (difference between the final output value and the step input),
- a higher overshoot.

Especially because of the last property one cannot increase the proportional

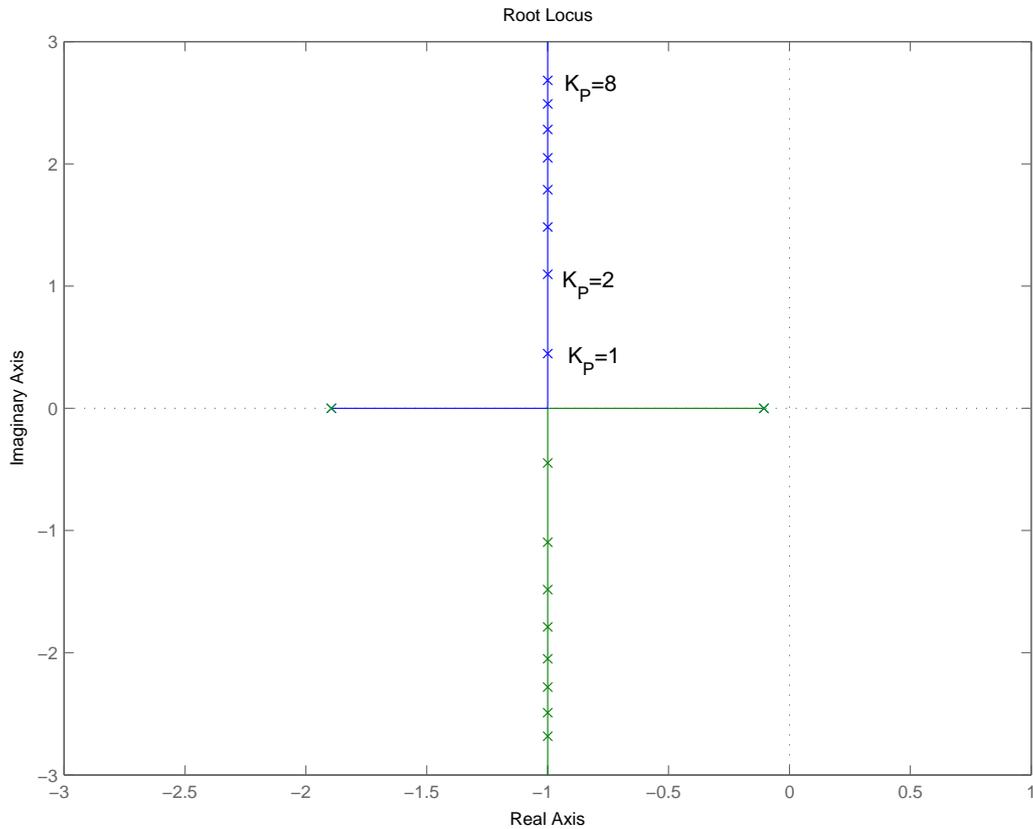


Figure 4: Root locus for the system (24) with $a_1 = 2$, $a_0 = 0.2$

gain K_P arbitrarily. Instead of that one adds an integral or a differential element to the control law (or both). The main advantage of the integral element is its capability to cancel any steady state error, while it increases the overshoot. The derivative or D-part of the controller is able to decrease the overshoot, while it is not affecting the steady-state error significantly. Therefore a commonly used control architecture is the so called PID controller, which is formed by the sum of a proportional an integral and a derivative element. The transfer function of the PID controller is:

$$C(s) = \frac{u}{v} = \frac{K_D s^2 + K_P s + K_I}{s} \quad (26)$$

Consequently the transfer function of the closed loop second order system

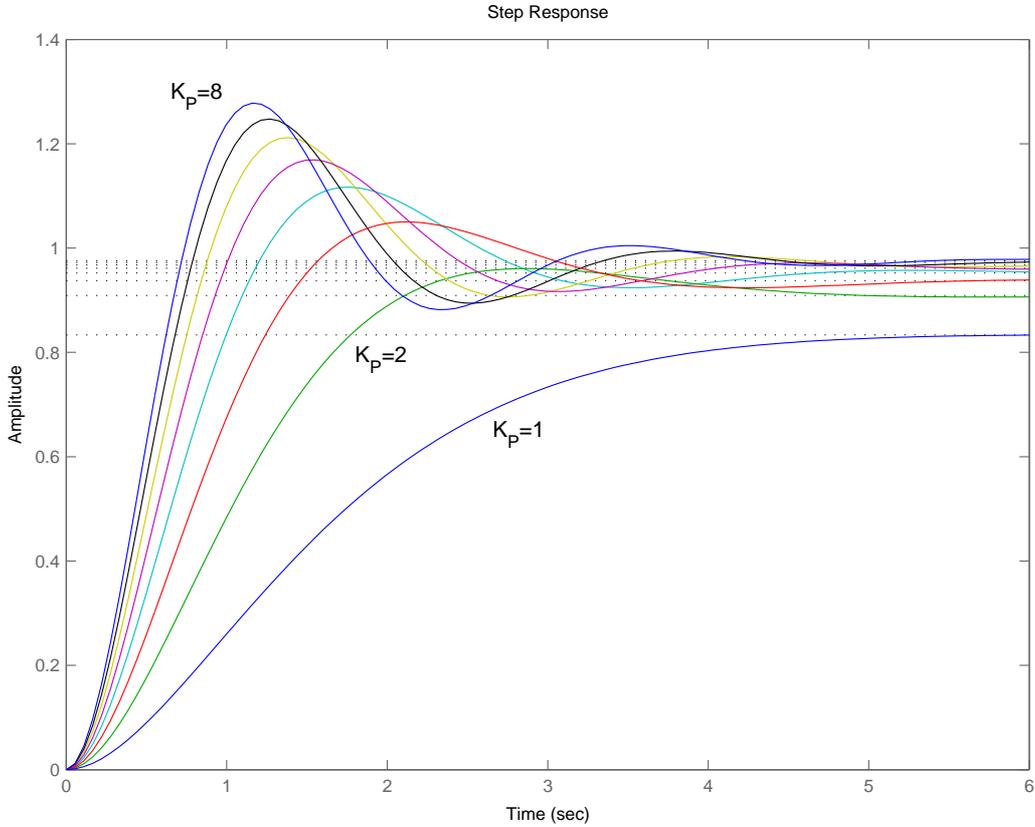


Figure 5: Step response for the system (24) with $a_1 = 2, a_0 = 0.2$

becomes:

$$\frac{y}{r} = \frac{K_D s^2 + K_P s + K_I}{s^3 + (a_1 + K_D) s^2 + (a_0 + K_P) s + K_I} \quad (27)$$

From (27) the transfer function of the PD controlled system can be derived by setting $K_I = 0$. In that case one would get:

$$\frac{y}{r} = \frac{K_D s + K_P}{s^2 + (a_1 + K_D) s + a_0 + K_P} \quad (28)$$

In the PD and PID controlled case it is possible to achieve any desired pole configuration of the closed loop system, because except of the coefficient in front of the highest order term all coefficients of the denominator polynomial are independently selectable by the parameters of the controller!

A possible problem of the above introduced PID controller lies in the availability of the used signals. This can be understood by writing the control

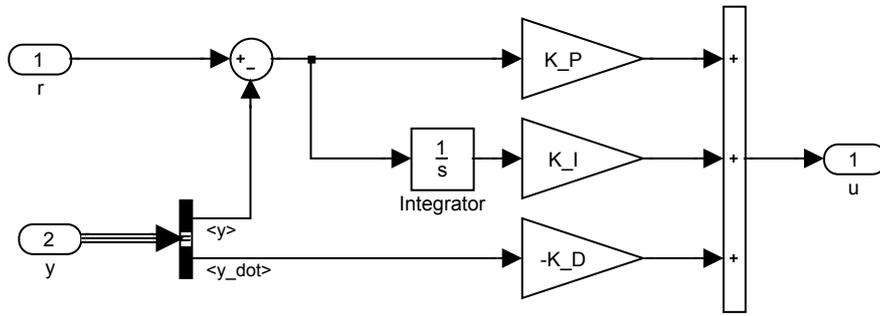
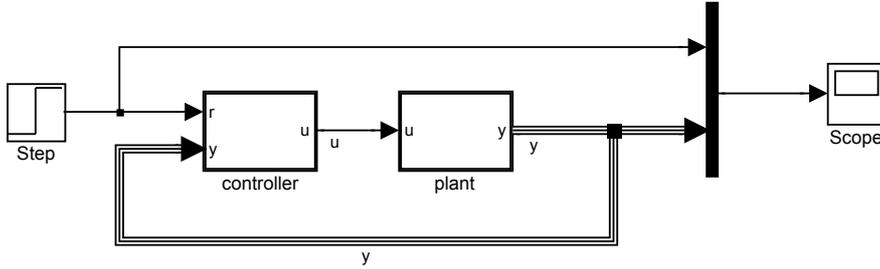


Figure 7: Block diagram of the controller in Figure 6

equation (20) for the PID controller in the time domain. There it becomes:

$$u(t) = K_D \frac{d}{dt}(r - y) + K_P(r - y) + K_I \int_0^t (r(\tau) - y(\tau)) d\tau \quad (29)$$

For practical application the differentiation of a measured signal ($r - y$) should be avoided. By assuming that y and \dot{y} are directly measurable one could use the following control law instead:

$$u(t) = -K_D \dot{y} + K_P(r - y) + K_I \int_0^t (r(\tau) - y(\tau)) d\tau \quad (30)$$

Applying this control law, one gets for the closed loop transfer function:

$$\frac{y}{r} = \frac{K_P s + K_I}{s^3 + (a_1 + K_D) s^2 + (a_0 + K_P) s + K_I} \quad (31)$$

The related block diagrams are shown in Figure 6 and Figure 7.

6 Homework

1. What kind of controller is required for being able to freely chose the poles of the closed loop system, if the plant dynamics are given by:

$$\dot{y} + ay = u$$

2. Between the Laplace space and the time domain, the following relation holds:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Use this relation to calculate the final value of the PD and the PID controlled system upon a step input ($r(s) = \frac{1}{s}$)

3. In order to achieve steady state accuracy with a PD controller, someone suggests to use the control law:

$$u = (K_D s + K_P)(k_v r - y)$$

How has the gain k_v to be chosen for that purpose? How can the block diagram (Figure 3) be modified accordingly?

4. Given the second order plant (23) with $a_1 = 1, a_2 = 2$. Calculate the gains of a PID controller, such that the closed system has the following pole configuration:

$$s_1 = -2, s_2 = -1.5 - j, s_3 = -1.5 + j$$

5. Implement the system in Simulink. (use the step function as input)

7 How to chose the poles of the transfer function

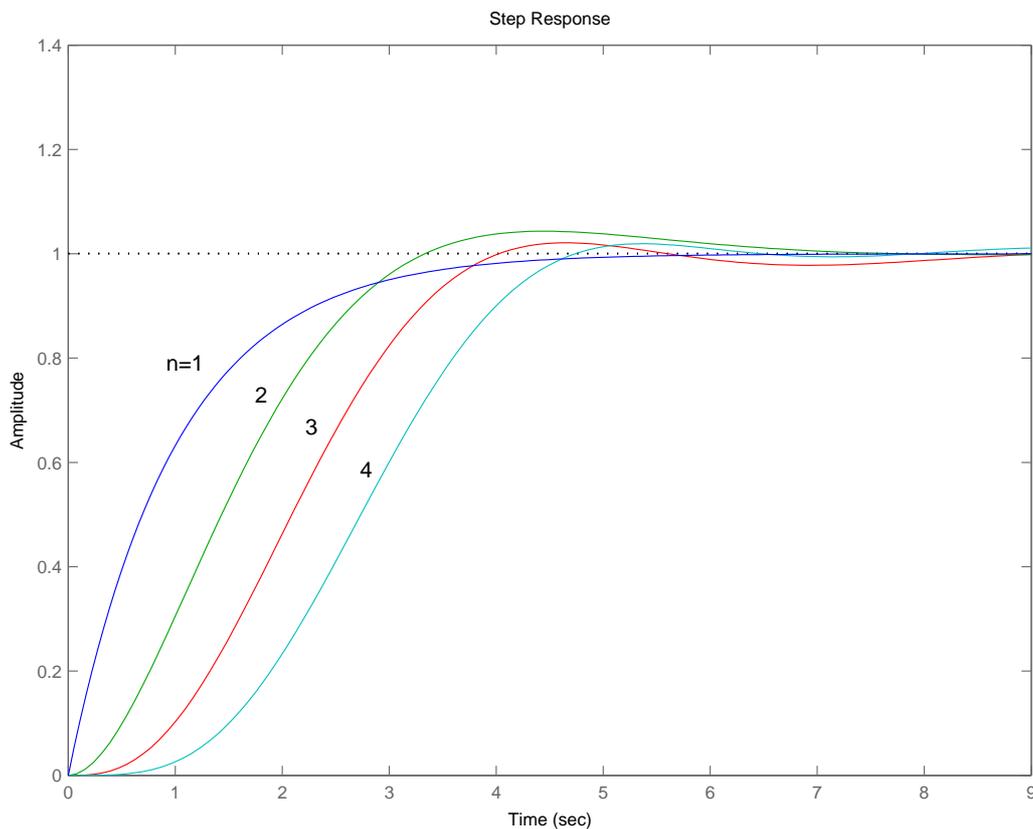


Figure 8: Step responses with *ITAE* pole configuration

The incorporation of a controller in the system allows to manipulate the pole configuration of the transfer function. In that context the question arises what pole configuration is desirable. A possible way for answering this question can be given by examining the influence of the pole configuration upon the step response of the system.

Very often the minimization of the *ITAE* (Integral Time multiplied Absolute Error) criteria is used for determining the optimal pole configuration. The *ITAE* criteria is defined as:

$$I_{ITAE} = \int_0^{\infty} t|e(t)|dt \quad (32)$$

The optimal step responses with respect to the *ITAE* criteria is achieved by the following pole configurations:

system order n	basis pole configuration
1	-1
2	$-0.7071 \pm 0.7071j$
3	$-0.7071 \quad -0.521 \pm 1.068j$
4	$-0.424 \pm 1.263j \quad -0.626 \pm 0.4141j$

The related step responses can be seen in Figure 8.

8 Frequency response characteristic

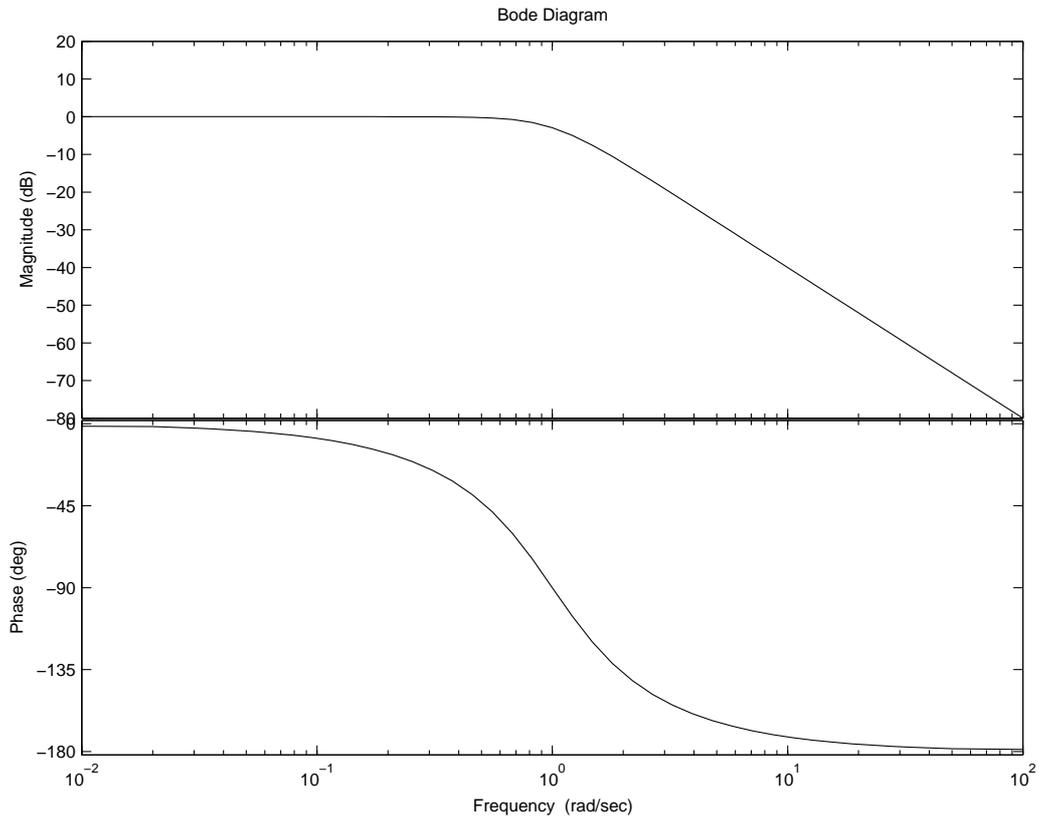


Figure 9: `bode([1],[1 1.4 1])`

Another important tool in the analysis of a dynamical system is the frequency response characteristic, which describes the output or response of a

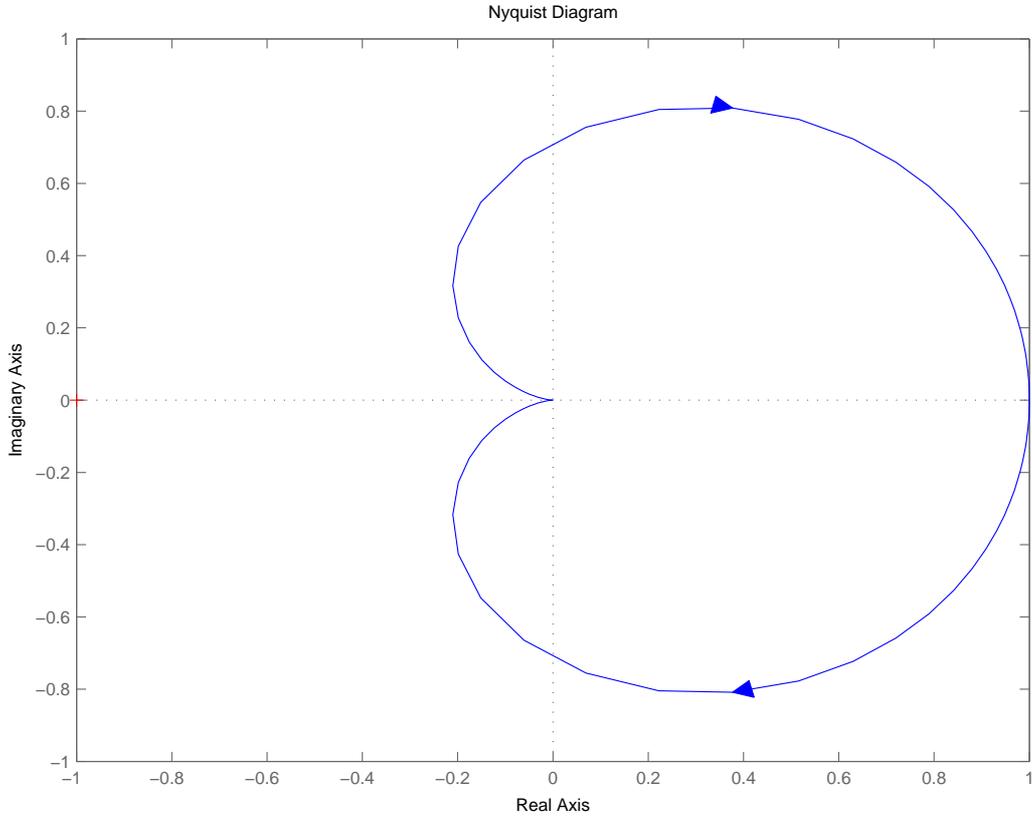


Figure 10: $\text{nyquist}([1],[1 \ 1.4 \ 1])$

system upon an harmonic input. From the theory of linear differential equations it is known, that the partial solution of harmonically disturbed linear system:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y} + a_0y = \cos(\omega t) = \text{Re} \{ e^{j\omega t} \} \quad (33)$$

is given by an oscillation of the same angular frequency. Hence for the partial solution one can use the approach:

$$y_p = b_1 \cos(\omega t) + b_2 \sin(\omega t) = \text{Re} \{ \mathbf{d}e^{j\omega t} \} \quad (34)$$

with $\mathbf{d} \in \mathbb{C}$. The relation between the coefficients b_1, b_2 and the complex

vector \mathbf{d} is:

$$\mathbf{d} = e^{j\phi} \sqrt{b_1^2 + b_2^2} \quad (35)$$

$$\phi = -\arctan\left(\frac{b_2}{b_1}\right) \quad (36)$$

The frequency response characteristic of a system describes the dependence of the vector \mathbf{d} upon the angular frequency ω of the input. The calculation of \mathbf{d} can be done by inserting (34) into (33):

$$\begin{aligned} \operatorname{Re}\{\mathbf{d}(j\omega)^n e^{j\omega t} + a_{n-1}\mathbf{d}(j\omega)^{(n-1)}e^{j\omega t} + \dots + a_1\mathbf{d}j\omega e^{j\omega t} + a_0\mathbf{d}e^{j\omega t}\} &= \operatorname{Re}\{e^{j\omega t}\} \\ \Rightarrow \mathbf{d} &= \frac{1}{(j\omega)^n + a_{n-1}(j\omega)^{(n-1)} + \dots + a_1j\omega + a_0} \end{aligned} \quad (37)$$

Equation (37) is normally referred to as the frequency response characteristic (FRC) of the system:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y} + a_0y = u \quad (38)$$

Interestingly the FRC of a linear system can be achieved directly from its transfer function by replacing s by $j\omega$!

There are mainly two ways for visualizing the vector \mathbf{d} :

- The so called *Bode* diagram, where the amplitude and the phase angle of \mathbf{d} are plotted separately.
- The so called *Nyquist* diagram, where \mathbf{d} is directly plotted in the complex plane.

Figure 9 shows the Bode diagram for the linear system with the transfer function:

$$G(s) = \frac{Y}{U} = \frac{1}{s^2 + 1.4s + 1} \quad (39)$$

The Nyquist diagram for the same system can be seen in Figure 10.