On Independent Spanning Trees

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Abstract

We prove that if any k-vertex connected graph has k vertex independent spanning trees, then any k-edge connected graph has k edge independent spanning trees. Thus, answering a question raised by Zehavi and Itai [J. of Graph Theory, 13 (1989)] in the affirmative.

1. Introduction

Two spanning trees of a graph $G = (V, E)$ are called vertex (resp. edge) *independent* if they are rooted at the same vertex r, and for each vertex $v \in V$, the two paths from v to r, one path in each tree, are internally vertex (resp. edge) disjoint. The k spanning trees of G are said to be vertex (resp. edge) independent if they are pairwise independent.

Itai and Rodeh [IR], gave a linear time algorithm for finding two vertex independent spanning trees in a biconnected graph. They left open the problem of constructing k vertex independent spanning trees in a k-vertex connected graph for $k > 2$ (or even showing their existence). Subsequently, Cheriyan and Maheshwari [CM], and Zehavi and Itai [ZI] proved the conjecture for $k = 3$. The proof of [CM] is constructive and yields an algorithm with a running time of $O(n^2)$, where n is the number of vertices in G. The conjecture for arbitrary values of k is still open.

In their paper [ZI] state the following two "versions" of the k independent spanning trees conjecture.

Conjecture 1.1 (Vertex Conjecture): *Any k-vertex connected graph has k vertex independent spanning trees rooted at an arbitrary vertex r.*

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Conjecture 1.2 (Edge Conjecture): *Any k-edge connected graph has k edge independent spanning trees rooted at an arbitrary vertex r.*

Both the conjectures are currently open. Zehavi and Itai [ZI] raised the following question:

> *It would be interesting to show that either the vertex conjecture implies the edge conjecture, or vice-versa.*

In this note we show that using a technique similar to the one developed by Galil and Italiano [GI] (for reducing edge connectivity to vertex connectivity), it is possible to show that the vertex conjecture implies the edge conjecture.

2. Main Proof

Galil and Italiano [GI] showed a reduction from edge connectivity to vertex connectivity. Formally, given a graph G they showed how to convert it to a graph G' that is k-vertex connected iff G is k-edge connected. For our purposes it is more convenient to modify the reduction and work with a more "uniform" reduction.

Suppose that the vertex conjecture holds. We show that in this case the edge conjecture holds as well. Given a graph that is k-edge connected, we show how to construct k edge independent spanning trees. First, we apply a transformation to convert G to a graph G' that is k-vertex connected. By our assumption G' has k independent spanning trees. Then, we show how to generate k edge independent spanning in G using the k vertex independent spanning trees in G'. (This is done in Theorem 2.2.)

The Transformation:
Given a graph $G = (V, E)$, define the graph $G' = (V', E')$ as follows. For each vertex v of G, there are k vertices $v^1, v^2, \ldots, v^k$ in G'. These are called *node vertices* of G'. For each edge e of G, there is a vertex $\ell(e)$ in G'. These are referred to as the *arc vertices* of G'.

The edges of G' are defined as follows. Let v be a vertex of G, and let $e_0, e_1, \ldots, e_{d-1}$ be the edges adjacent to v. For each e_i $0 \leq i \leq d-1$, there are edges $(\ell(e_i), v^j)$ $1 \leq j \leq k$ (i.e., we have a complete bipartite graph between the arc vertices $\{\ell(e_0), \ldots \ell(e_{d-1})\}$ and the node vertices corresponding to v).

Theorem 2.1: *The graph G' is k-vertex connected if and only if G is k-edge connected.*

Proof: The *if* direction: Suppose that G is k-edge connected. We show that G' is k-vertex connected. Suppose that G' is not k-vertex connected; that is, there is a set S of $k-1$ vertices in G' whose removal disconnects G'. Notice that in this case each component must contain at least one node vertex. This is because at least k vertices have to be removed in order to disconnect a component consisting of only arc vertices. Suppose that u^i and v^j are disconnected after removing S. Consider the subgraph of G given by removing the edges corresponding to the arc vertices in S. By our assumption this subgraph is connected, and thus contains a path P between u and v. Since $|S| < k$, for each vertex w in P at least one node vertex corresponding to w is not in S. Denote each such vertex as $f(w)$. Also, none of the edges in P correspond to arc vertices in S. Let $u_0(= u), e_1, u_1, \ldots, e_l, u_l(= v)$ be the representation of P as an interleaving sequence of vertices and edges. It is not difficult to see that the sequence of vertices $u^i, \ell(e_1), f(u_1), \ldots, \ell(e_l), v^j$ is a path in the subgraph of G' given after removing S – a contradiction.

The *only if* direction: Suppose that G' is k-vertex connected. We show that G is k-edge connected. Suppose that G is not k-edge connected; that is, there is a set S of $k-1$ edges in G whose removal disconnects G. Suppose that u and v are disconnected after removing S. Consider the subgraph of G' given by removing the arc vertices corresponding to the edges in S. By our assumption this subgraph is connected, and thus contains a path P' between u^1 and v^1. Since G' is bipartite with arc vertices in one side and node vertices in the other, the path P' is interleaving between node and arc vertices. All the edges corresponding to arc vertices in P' are not in S. Thus, any two adjacent node vertices in P' correspond either to adjacent nodes or to the same node in the subgraph of G' given after removing S. This implies that the sequence of vertices in G that correspond to the node vertices in P', form a path from u to v in this subgraph – a contradiction.

□

Theorem 2.2: *Given k independent spanning trees in G' rooted at r^1, we can obtain k edge independent spanning trees in G rooted at r.*

To prove this theorem, we first describe the construction for the edge independent spanning trees in G and then prove that it is correct.

We will assume that the independent trees in G' are $T'_1, T'_2, \ldots T'_k$. Using these trees we will see how to generate trees $T_1, \ldots, T_k$ in G that are edge independent. For each vertex $v \in G$ (assuming that $v \neq r$) we need to define its parents in the k trees $T_1, \ldots, T_k$. We define $group(v)$ to be the k vertices in G' corresponding to vertex v in G. From v^1 there are k vertex disjoint paths going to the root (one path in each tree). Let us call these paths $P'_1[v^1, r^1], \ldots, P'_k[v^1, r^1]$.

The parent of vertex v in the tree T_j is defined as follows. Let $v^{f(j)}$ be the *last vertex* on the path $P'_j[v^1, r^1]$ that belongs to $group(v)$. (Clearly such a vertex exists since v^1 is in $group(v)$ and r^1 is not.) Let the outgoing edge from $v^{f(j)}$ on P'_j be $(v^{f(j)}, \ell(e_m))$, for $e_m = (v, u)$ in G. Then, the parent of v in T_j is defined to be u.

Once we define the parents for each vertex (other than the root) in each of the trees, this fully specifies the k trees. Observe that this operation will yield k paths in G from each vertex v to r. The paths $P_1, \ldots, P_k$ in G' were vertex disjoint, and essentially we did a "shortcutting" operation on the paths. The "shortcutting" step achieves the effect of making all the paths from a vertex to the root into simple paths. The paths do not remain vertex disjoint anymore due to paths using different vertices that belong to the same group (these shrink to a single node in G). We now prove that the paths are edge disjoint.

Lemma 2.3: *The paths $P_1, \ldots, P_k$ from v to r that are the unique paths to r in each tree are edge disjoint.*

Proof: Assume that there are two paths $P_1[v, r]$ and $P_2[v, r]$ that use the same edge e. This implies that both paths $P_1'[v^1, r^1]$ and $P_2'[v^1, r^1]$ use the same vertex $\ell(e)$, contradicting the assumption that the paths $P_1'[v^1, r^1]$ and $P_2'[v^1, r^1]$ are internally vertex disjoint. □

This concludes the proof of Theorem 2.2.

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