Asymptotic entanglement capacity of the Ising and anisotropic Heisenberg interactions

Andrew Childs
MIT Center for Theoretical Physics
IBM T. J. Watson Research Center

Debbie Leung
IBM T. J. Watson Research Center

Frank Verstraete
SISTA/ESAT, University of Leuven

Guifré Vidal
Caltech Institute for Quantum Information

arxiv.org/abs/quant-ph/0207052

Outline

- Entanglement as a resource
- Capacities of interactions to produce entanglement
- Two-qubit Hamiltonians: the canonical form
- Capacity of $\mu_x\, X \otimes X + \mu_y\, Y \otimes Y$
- Numerical results
- Open problems

Resources in (quantum) information theory

Information is a resource.

- Physical
- Fungible

Examples for two-party problems:

	Static	Dynamic
Classical	cbits$_{A\to B}$ cbits$_{B\to A}$ sbits	Channel
Quantum	qubits$_{A\to B}$ qubits$_{B\to A}$ ebits	Unitary gate Hamiltonian Quantum operation

Quantum information theory is about the interconversion of informational resources.

What is entanglement?

Entangled pure state:

$$|\psi\rangle_{AB} \neq |\phi\rangle_A |\eta\rangle_B$$

Canonical example: EPR pair

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

Entanglement = non-classical correlations

- Violation of Bell inequalities
- Can be used to perform classically impossible tasks!

The many uses of entanglement

- Superdense coding [Bennett, Wiesner 92]
- Quantum teleportation [Bennett et al. 93]
- Quantum key distribution [Lo, Chau 98]
- Entanglement-assisted classical communication
  ... through unidirectional channels [Shor et al. 99]
  ... through bidirectional channels [Bennett et al. 02]
- Remote state preparation [Lo 00, Bennett et al. 00]
- Data hiding [DiVincenzo et al. 00]
- Quantum Vernam cipher [Leung 00]

⋮

Quantifying entanglement

Consider a bipartite state $|\psi\rangle$.

Any such state has a Schmidt decomposition:

$$|\psi\rangle = \sum_j \sqrt{p_j} |j\rangle_A |\tilde{j}\rangle_B$$

where $\sum_j p_j = 1$ and $\{|j\rangle_A\}$, $\{|\tilde{j}\rangle_B\}$ are orthonormal bases.

Entanglement:

$$E(|\psi\rangle) = -\sum_j p_j \log p_j$$

measured in *ebits*.

$$1 \text{ ebit} = E(|\Psi^+\rangle)$$

Entanglement is fungible

Theorem. Asymptotically, states with the same entanglement are interconvertible.

[Bennett et al. 95]

Entanglement concentration

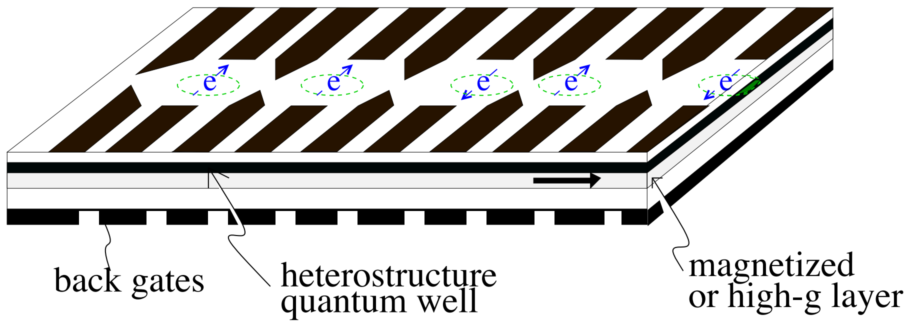
$$n \text{ copies of } |\psi\rangle \xrightarrow{\text{LO}} nE(|\psi\rangle) \text{ ebits}$$

Entanglement dilution

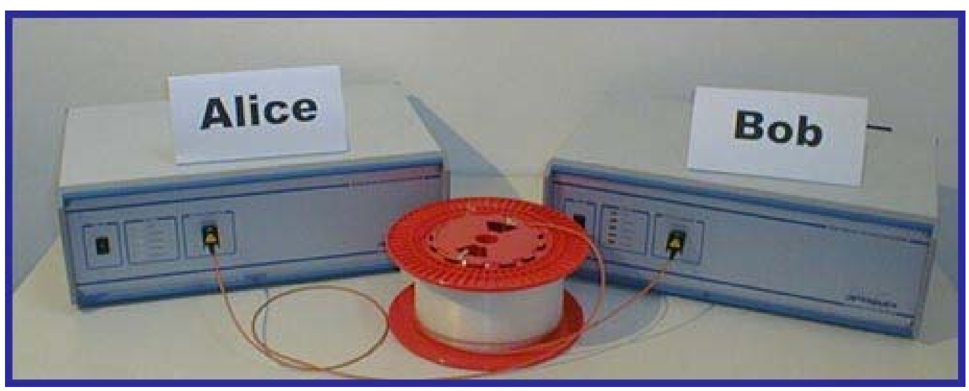
$$nE(|\psi\rangle) \text{ ebits} \xrightarrow{\text{LOCC}} n \text{ copies of } |\psi\rangle$$

Physical systems for entanglement generation

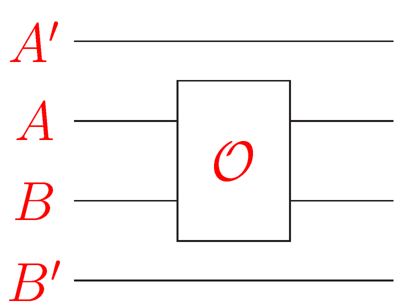
- Adjacent quantum dots



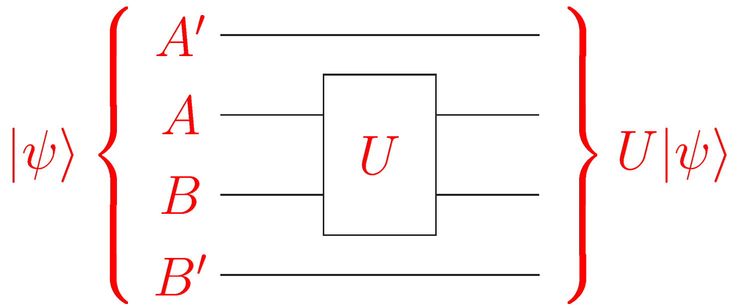
- Distant labs connected by optical fiber



General model:



How to make entanglement



Choose $|\psi\rangle$ so that $U|\psi\rangle$ is more entangled than $|\psi\rangle$.

Entanglement generating capacity

$$E_U = \sup_{|\psi\rangle \in AA'BB'} \left[E(U|\psi\rangle) - E(|\psi\rangle) \right]$$

Three technical points:

- Ancillary systems
- Mixed states
- Asymptotic vs. one-shot capacity

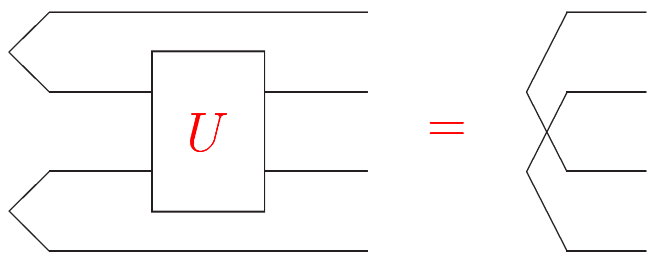
Using ancillas

Consider $U = \mathrm{SWAP}$:

$$U|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle$$

Clearly $E(|\psi\rangle_{AB}) = E(U|\psi\rangle_{AB})$.

But:



In general, you can make more entanglement when ancillary systems are available. This makes it hard to compute E_U!

Mixed states

> **Theorem.** For unitary interactions, the optimal input state is always pure.
>
> [Bennett, Harrow, Leung, Smolin 02]

Proof:

$$
\begin{aligned}
E'_U &= \sup_\rho [D(U\rho U^\dagger) - E_c(\rho)] \\
&\leq \sup_\rho [E_c(U\rho U^\dagger) - E_c(\rho)] \\
&= \sup_\rho \frac{1}{n}[E_f((U\rho U^\dagger)^{\otimes n}) - E_f(\rho^{\otimes n})] + \epsilon \\
&= \sup_\rho \frac{1}{n}\sum_i p_i [E((U|\psi_i\rangle)^{\otimes n}) - E(|\psi_i\rangle^{\otimes n})] + \epsilon \\
&= \sup_\rho \sum_i p_i [E(U|\psi_i\rangle) - E(|\psi_i\rangle)] \\
&= \sup_{\rho,i} [E(U|\psi_i\rangle) - E(|\psi_i\rangle)] \\
&= E_U
\end{aligned}
$$

□

Asymptotic vs. one-shot

Theorem. $E_U^{(n)} = nE_U$

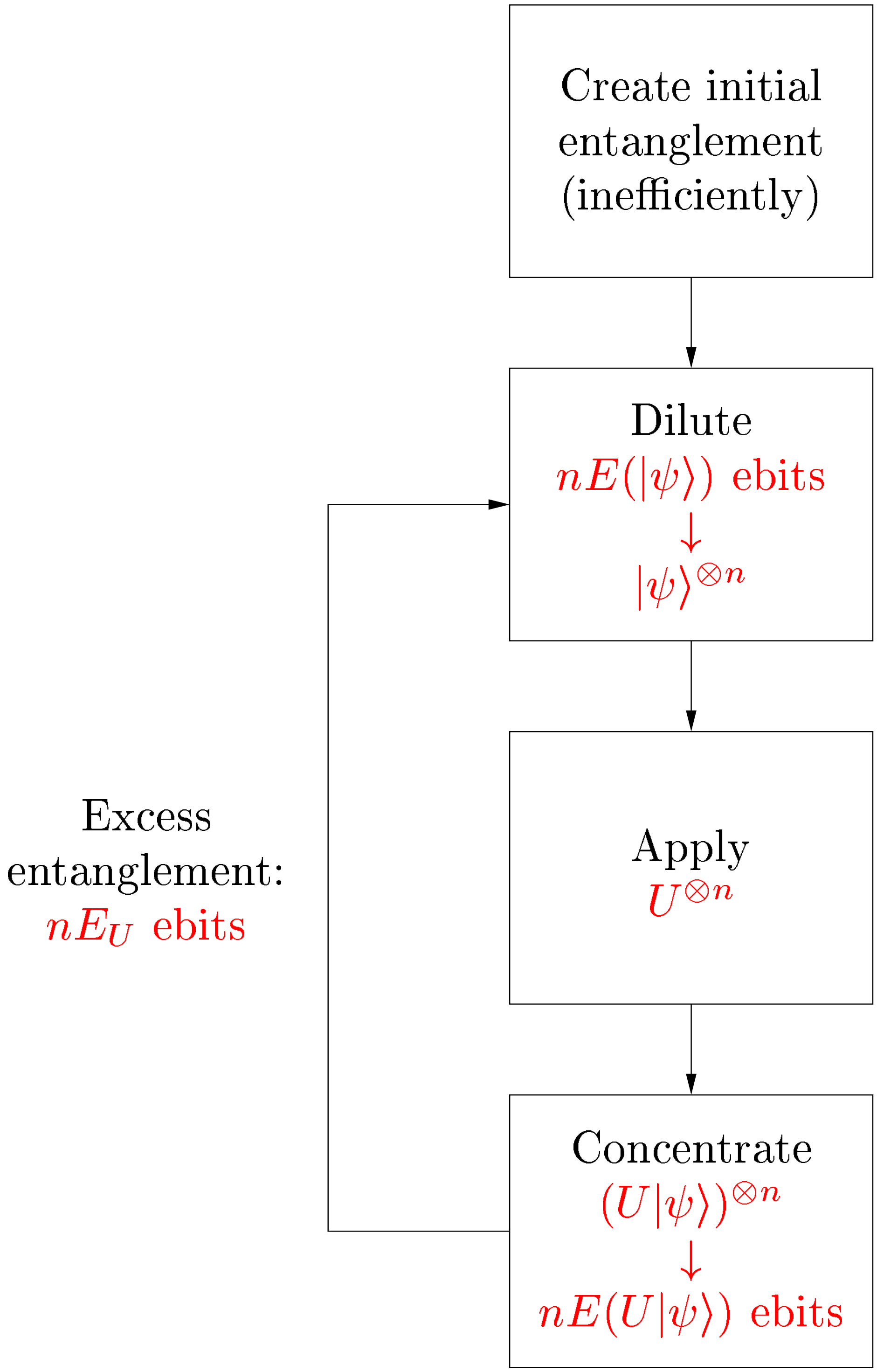
[Bennett, Harrow, Leung, Smolin 02]

Proof:

The entanglement can only increase by application of U.
For each use of U, the maximum increase is given by E_U.
Thus $E_U^{(n)} \leq nE_U$.

By using the optimal input n times, $E_U^{(n)} \geq nE_U$. $\square$

Entanglement production cycle



Entanglement capacity of a Hamiltonian

$$
\begin{aligned}
E_H &= \lim_{t\to 0}(E_{e^{-iHt}}/t) \\
&= \sup_{|\psi\rangle}\left[\frac{\mathrm{d}}{\mathrm{d}t}E(e^{-iHt}|\psi\rangle)\right]_{t=0}
\end{aligned}
$$

Using perturbation theory, we find

$$
E_{H,|\psi\rangle} = \sum\nolimits_{j,k} \sqrt{p_j p_k}\log(p_j/p_k)\ \operatorname{Im}\langle j\tilde{j}|H|k\tilde{k}\rangle
$$

where

$$
|\psi\rangle = \sum_j \sqrt{p_j}|j\rangle_{AA'}|\tilde{j}\rangle_{BB'}
$$

This is...

- Zero for product states
- Zero for maximally entangled states
- Hard to optimize over $|\psi\rangle$!

Two-qubit Hamiltonians: Canonical form

A general two-qubit Hamiltonian has 16 real parameters. But only two of them are nonlocal!

Fact: Any two-qubit Hamiltonian H is *locally equivalent* to a Hamiltonian of the form

$$\tilde{H} = \mu_x \, X \otimes X + \mu_y \, Y \otimes Y + \mu_z \, Z \otimes Z \, .$$

In other words, there are local Hamiltonians H_A, H_B and local unitary operators U, V so that

$$H = (U \otimes V) \tilde{H} (U^\dagger \otimes V^\dagger) + H_A + H_B \, .$$

[Dür et al. 01]

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Ising interaction

Consider $H = Z \otimes Z$ $\qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Note that H is locally equivalent to $2|00\rangle\langle 00|$.

No ancillas:

$$\begin{aligned} E^*_{ZZ} &= 4 \max_{p,|\psi\rangle} \sqrt{p(1-p)} \log \frac{p}{1-p} \operatorname{Im}(\langle\psi|00\rangle\langle 00|\psi^\perp\rangle) \\ &= 2 \max_{p} \sqrt{p(1-p)} \log \frac{p}{1-p} \\ &\approx 1.9123 \end{aligned}$$

[Dür et al. 01]

Theorem. $E_{ZZ} = 1.9123$

[Childs, Leung, Vidal, Verstraete 02]

Proof idea: No pair of terms in the Schmidt decomposition with Schmidt coefficients p_1, p_2 can contribute more than $E^*_{ZZ}/(p_1+p_2)$. □

$\mu_x \, XX + \mu_y \, YY$

Upper bound: Simulation.

The Hamiltonian $\mu_x \, X \otimes X + \mu_y \, Y \otimes Y$ can be *simulated* using $(\mu_x + \mu_y) \, Z \otimes Z$.

- There exist unitaries H, K so that

$$HZH^\dagger = X \qquad\qquad KZK^\dagger = Y$$

- Use the Lie product formula

$$e^{-i(H_1+H_2)t} = \lim_{n\to\infty} (e^{-iH_1 t/n} e^{-iH_2 t/n})^n$$

Therefore $E_{\mu_x XX + \mu_y YY} \leq (\mu_x + \mu_y) E_{ZZ}$.

Lower bound: By the explicit protocol (with no ancillas), $E_{\mu_x XX + \mu_y YY} \geq (\mu_x + \mu_y) E_{ZZ}$. [Dür et al. 01]

Summary of known capacities

Gates:

$$E_{\mathrm{CNOT}} = 1$$
$$E_{\mathrm{SWAP}} = 2$$

Hamiltonians:

$$E_{\mu_x XX + \mu_y YY} = 1.9123(\mu_x + \mu_y)$$

In general, there may be no closed form expression for the capacity of a given interaction.

For the Hamiltonian

$$H = \mu_{xy}(X \otimes X + Y \otimes Y) + Z \otimes Z$$

we conjecture

$$\begin{aligned} E_{\mu_{xy}(XX+YY)+ZZ} = 2 \max \{ & \sqrt{p_1 p_2} \log(p_1/p_2) \left[\sin\theta + \mu_{xy} \sin(\varphi - \xi)\right] \\ & + \sqrt{p_2 p_4} \log(p_2/p_4) \left[\sin\varphi + \mu_{xy} \sin(\theta - \xi)\right] \\ & + \sqrt{p_1 p_4} \log(p_1/p_4)\, \mu_{xy} \sin\xi \ \} \end{aligned}$$

where $p_1, p_2, p_4 > 0$, $p_1 + 2p_2 + p_4 = 1$, and $\theta, \varphi, \xi \in [0, 2\pi)$.

Open problems

- Calculate capacities for two-qubit gates
- Find an upper bound on the optimal ancilla dimension for a $d_A \times d_B$ dimensional gate or Hamiltonian
- Study entanglement generation by nonunitary quantum operations
- Inverse problem: How much entanglement is needed to simulate a gate (or Hamiltonian)?

  $$E_U \leq \text{ebits needed to simulate } U$$

  When is this achievable?