Spatial search and the Dirac equation

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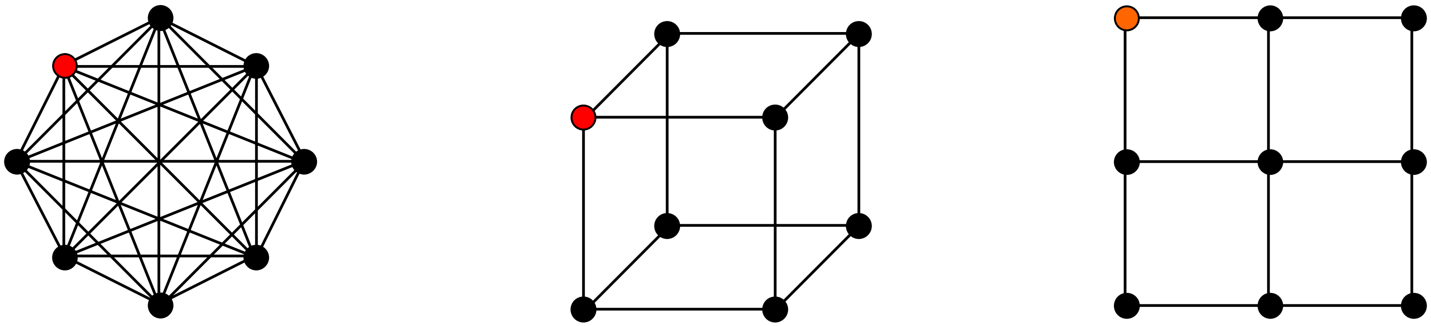


Find more quantum algorithms!

Fourier sampling

- Factoring, discrete log
- Hidden subgroup problems
- Pell’s equation
- Hidden shift problems

Amplitude amplification

- Unstructured search
- Constant-depth AND-OR trees
- Various graph problems

Quantum walk

- Exponential speedup for a black box problem
- Spatial search
- Element distinctness (→ triangle finding, verifying matrix product, etc.)

Unstructured search

N items {1,2,…,*N*}

Find one "marked item" *w*

Query: "is *w*=*x*?"

I.e., black box function $f(x) = \begin{cases} 0 & x \neq w \\ 1 & x = w \end{cases}$

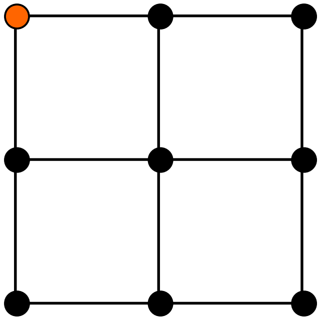
Classical complexity: Θ(*N*)

Grover 1996: O($N^{1/2}$) quantum algorithm

BBBV 1996: This is optimal

Grover searching can be applied to a wide variety of other problems. But can it be used to search a physical database, where the *N* items are distributed in space?

Spatial search



Suppose the N items are the vertices of a graph, and the algorithm is restricted to access them by local moves along edges.

(Benioff 00: “Quantum robot”)

Geometry matters.

Example: If the items are arranged on a line, no speedup is possible.

Two (essentially equivalent) models

1. Local Hamiltonian with a marking term $-|w\rangle\langle w|$; measure complexity in terms of time.
2. Alternate between queries and local unitary transformations; measure complexity in terms of total number of steps.

Searching a *d*-dimensional space

Naive implementation of Grover: Each reflection about a uniform state ("inversion about average") takes time $N^{1/d}$ (radius of database), and there are $N^{1/2}$ such steps.

Running time $O(N^{1/2+1/d})$.

Aaronson, Ambainis 03: Carefully optimized recursive search of subcubes using amplitude amplification.

$d>2$: $O(N^{1/2})$
$d=2$: $O(N^{1/2} \log^2 N)$

But do we really need such a complicated algorithm? In particular:

- Does the quantum robot need a memory whose size grows with N?
- Does it need to take different actions at different (non-marked) locations? At different times?

Also, can we do better when $d=2$?

Quantum walk algorithms

Can we search a region of space using homogeneous, time-independent dynamics?

Two possibilities:

- Continuous-time quantum walk
- Discrete-time quantum walk (needs a “coin”)

Results

Simple continuous-time walk: $d>4$	[CG03]
Discrete-time walk with appropriate “coin”: $d>2$	[AKR04]
Continuous-time walk with spin: $d>2$	[CG04]

Spatial search by quantum walk

Simplest algorithm: $H = -\gamma\, \underset{\uparrow}{L} - \underset{\uparrow}{|w\rangle}\langle w|$

Laplacian marked site

$$L_{ab} = \begin{cases} 1 & ab \in G \\ -\text{deg}(a) & a = b \\ 0 & \text{otherwise} \end{cases}$$

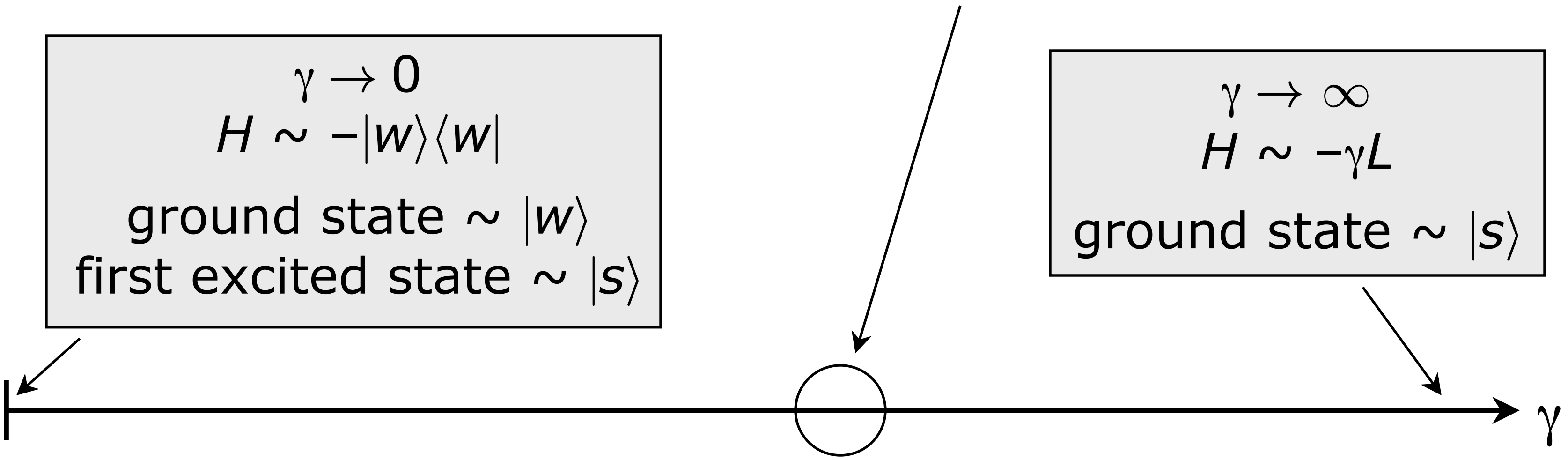
$\approx \nabla^2$

Start in $|s\rangle = \frac{1}{\sqrt{N}} \sum_{x \in G} |x\rangle$.

Choose some constant γ such that for as small a T as possible, $|\langle w|e^{-iHT}|s\rangle|^2$ is large.

critical γ

ground state $\sim |s\rangle + |w\rangle$
first excited state $\sim |s\rangle - |w\rangle$
time $\sim 1/(E_1 - E_0)$

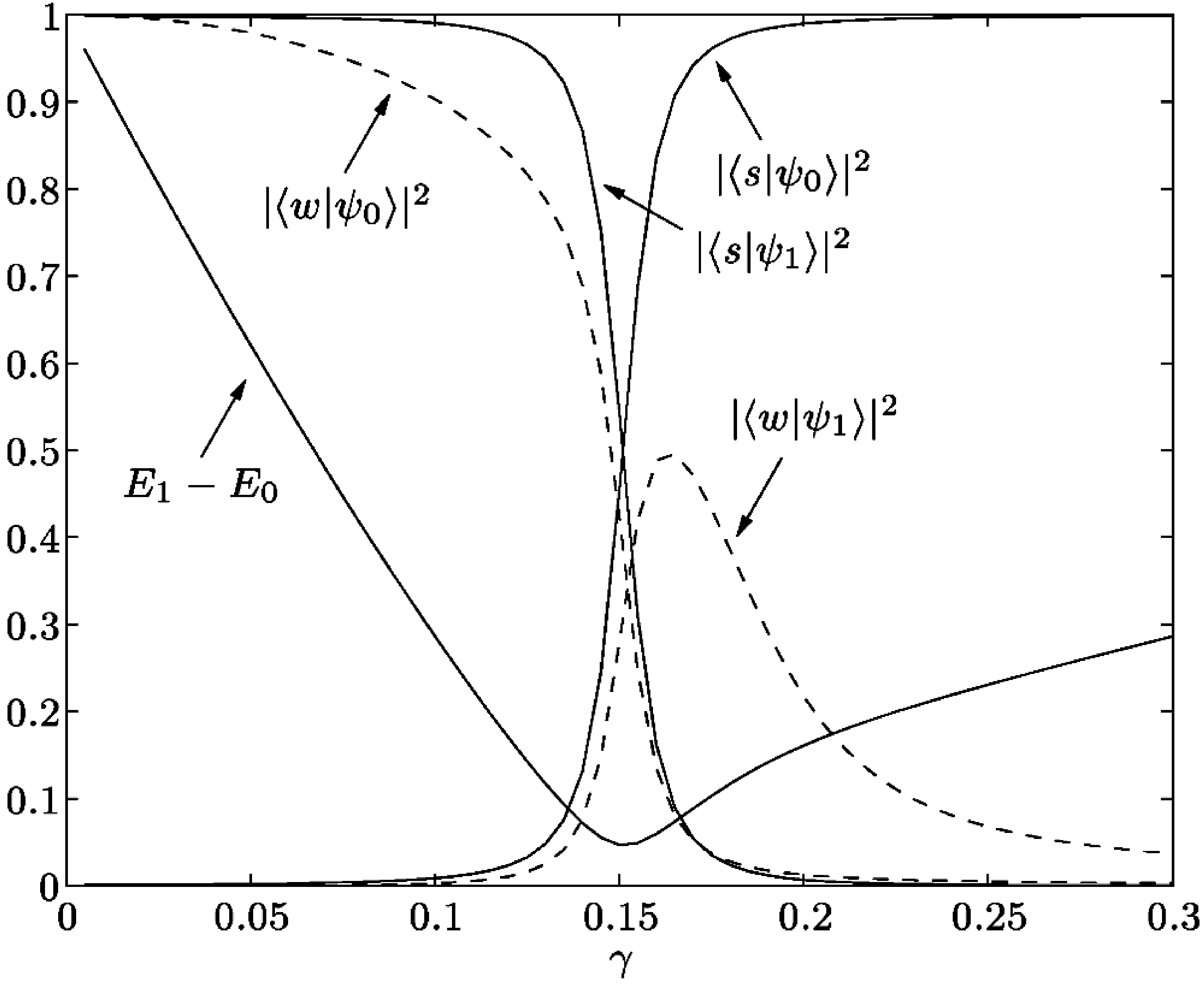


$\gamma \to 0$
$H \sim -|w\rangle\langle w|$

ground state $\sim |w\rangle$
first excited state $\sim |s\rangle$

$\gamma \to \infty$
$H \sim -\gamma L$

ground state $\sim |s\rangle$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
γ
$|\langle w|\psi_0\rangle|^2$
$|\langle s|\psi_0\rangle|^2$
$|\langle s|\psi_1\rangle|^2$
$|\langle w|\psi_1\rangle|^2$
$E_1 - E_0$

d=4, N=6^4=1296

Analysis

Use eigenstates of *L* to find eigenstates of *H*.

Periodic cubic lattice with *N* sites, size $N^{1/d}$ in each dimension.
Exact eigenstates and eigenvalues of $-\gamma L$:

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} |\vec{x}\rangle \qquad \mathcal{E}(\vec{k}) = 2\gamma \left(d - \sum_{j=1}^{d} \cos k_j \right)$$

$$k_j = \frac{2\pi m_j}{N^{1/d}}$$

$$m_j = \begin{cases} 0, \pm 1, \ldots, \pm\frac{1}{2}(N^{1/d} - 1) & N^{1/d} \text{ odd} \\ 0, \pm 1, \ldots, \pm\frac{1}{2}(N^{1/d} - 2), +\frac{1}{2}N^{1/d} & N^{1/d} \text{ even} \end{cases}$$

Results of analysis

Graph	**Success amplitude**	**Run time**
Complete	$1-o(1)$	$O(N^{1/2})$
Hypercube	$1-o(1)$	$O(N^{1/2})$
Lattice, $d>4$	$O(1)$	$O(N^{1/2})$
Lattice, $d=4$	$O(1/\log^{1/2} N)$	$O((N \log N)^{1/2})$
Lattice, $d=3$	$O(N^{-1/6})$	$O(N^{2/3})$
Lattice, $d=2$	$O((\log N/N)^{1/2})$	$O(N/\log N)$

Results of analysis, *d*>4

Critical γ: $\gamma_* = I_{1,d}$

Optimal run time: $T = \frac{\pi\sqrt{I_{2,d}N}}{2I_{1,d}}$

Success probability: $|\langle w|e^{-iHT}|s\rangle|^2 = \frac{I_{1,d}^2}{I_{2,d}}$

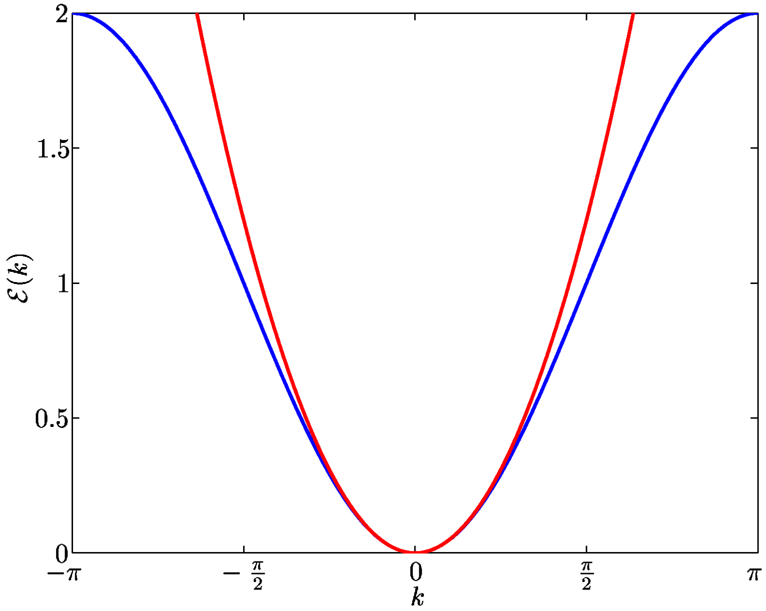
where

$$I_{j,d} = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\frac{\mathrm{d}^d\vec{k}}{[\mathcal{E}(\vec{k})]^j}$$

Note $\int_0 \frac{\mathrm{d}^d\vec{k}}{|\vec{k}|^p} \sim \int_0 \frac{|\vec{k}|^{d-1}\mathrm{d}k}{|\vec{k}|^p}$

converges for *d*>*p*.

dispersion relation



The Dirac equation

Hamiltonian: $$H_{\text{Dirac}} = \sum_{j=1}^{d} \alpha_j p_j + \beta m \qquad \vec{p} = -i\frac{\mathrm{d}}{\mathrm{d}\vec{x}}$$

where $\{\alpha_j, \alpha_k\} = 2\delta_{jk}\,, \quad \{\alpha_j, \beta\} = 0\,, \quad \beta^2 = 1$

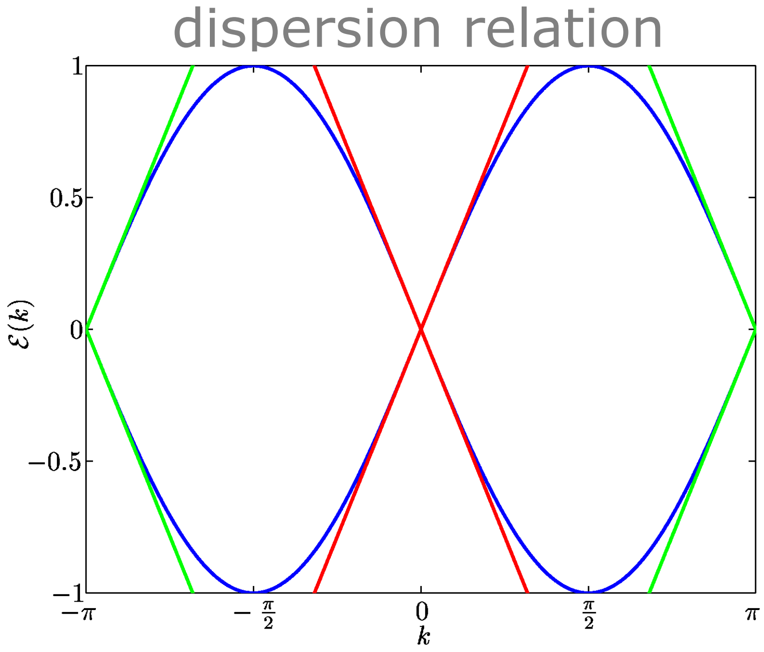
(number of spin components in *d* dimensions: $2^{\lceil d/2 \rceil}$)

Then $H^2_{\text{Dirac}} = |\vec{p}|^2 + m^2$, i.e. $E_{\text{Dirac}} = \pm|\vec{p}|$ for m=0.

Lattice version: $$H_0 = \omega \sum_{j=1}^{d} \alpha_j P_j$$

where $P_j|\vec{x}\rangle = \frac{i}{2}(|\vec{x} + \hat{e}_j\rangle - |\vec{x} - \hat{e}_j\rangle)$

$$\mathcal{E}(\vec{k}) = \pm\omega\sqrt{\sum_{j=1}^{d} \sin^2 k_j}$$



Discrete-time quantum walk search

Ambainis, Kempe, Rivosh 04: Discrete-time quantum walk search algorithm in d dimensions using a $2d$-dimensional “coin” space

Run times: $O(N^{1/2})$ for $d>2$, $O(N^{1/2} \log N)$ for $d=2$

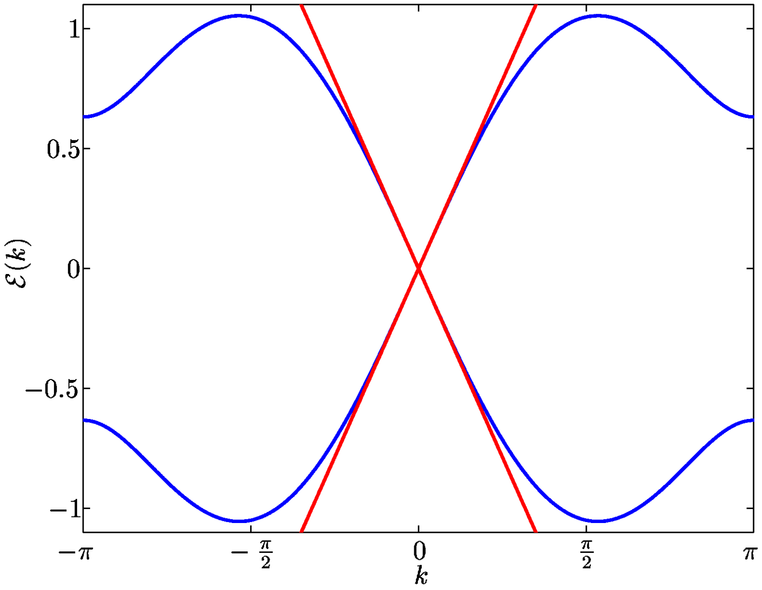
A discrete-time quantum walk cannot be defined on a state space consisting only of vertices (Meyer 96)

Making Dirac work
(or, Fixing fermion doubling)

Better lattice approximation: $H_0 = \omega \sum_{j=1}^{d} \alpha_j P_j + \gamma \beta L$

$$\mathcal{E}(\vec{k}) = \pm \sqrt{\omega^2 \sum_{j=1}^{d} \sin^2 k_j + \gamma^2 \Big[2 \sum_{j=1}^{d} (1 - \cos k_j) \Big]^2}$$

dispersion relation

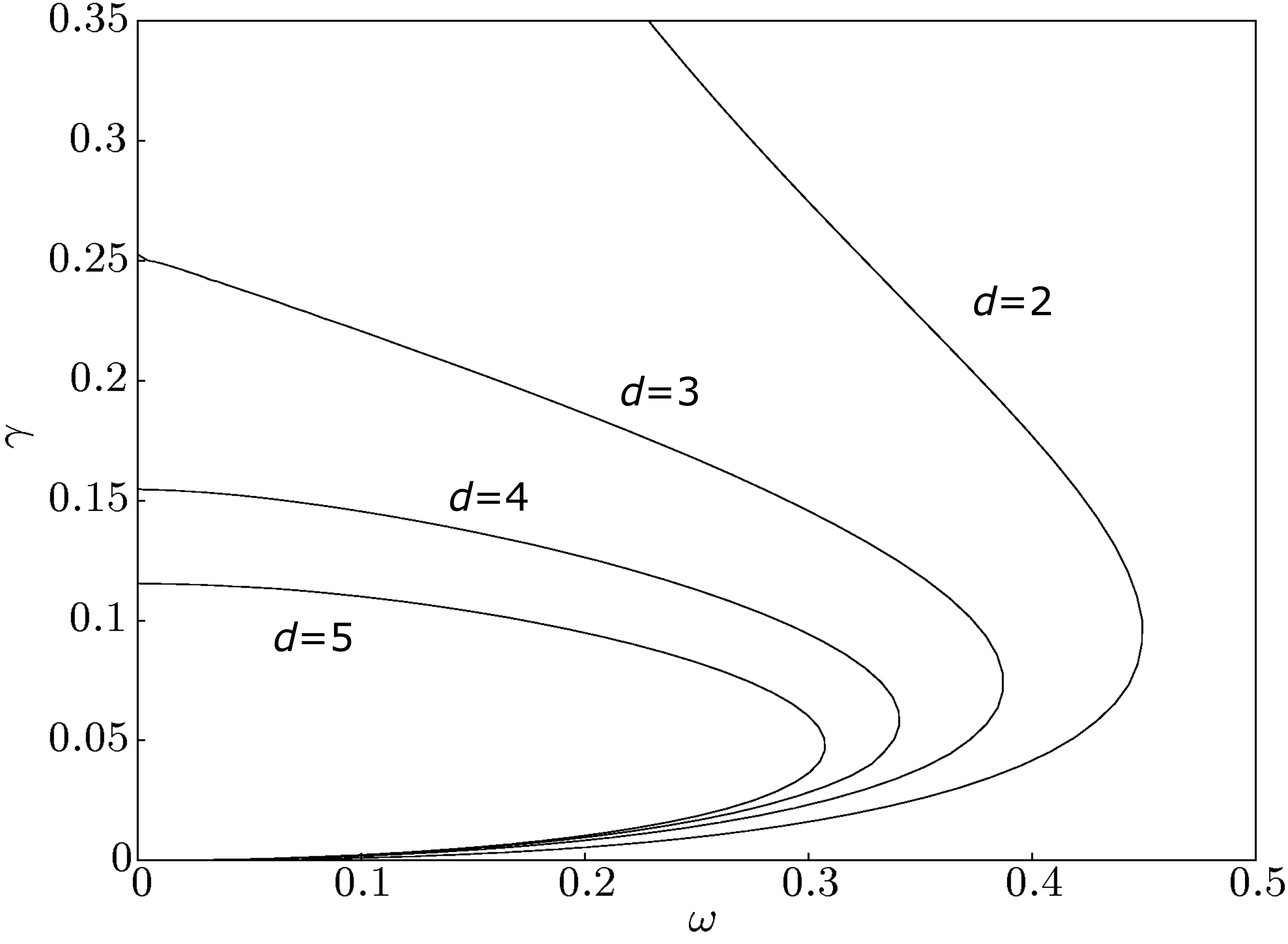


Algorithm:

Let $H = H_0 - \beta|w\rangle\langle w|$.

Start in $|\eta, s\rangle$.

Choose some constants ω, γ such that for as small a T as possible, $|\langle \eta, w| e^{-iHT} |\eta, s\rangle|^2$ is large.


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
γ
0
0.1
0.2
0.3
0.4
0.5
ω
d=2
d=3
d=4
d=5

Results of analysis (Dirac)

Graph	**Success amplitude**	**Run time**
Lattice, $d>2$	$O(1)$	$O(N^{1/2})$
Lattice, $d=2$	$O(1/\log^{1/2} N)$	$O((N \log N)^{1/2})$

Run time $O(N^{1/2} \log^{3/2} N)$ using classical repetition
$O(N^{1/2} \log N)$ using amplitude amplification

How many spin degrees of freedom?

Dirac particle in *d* dimensions: $2^{\lceil d/2 \rceil}$

(Smallest representation of Dirac algebra

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}\,, \quad \{\alpha_j, \beta\} = 0\,, \quad \beta^2 = 1$$

uses matrices of dimension $2^{\lceil d/2 \rceil}$.)

Simple continuous-time quantum walk: no spin!

Ambainis, Kempe, Rivosh discrete-time algorithm:
2*d* “coin states” (or 2 for *d*=2)

But it is sufficient to reproduce the action of the Dirac algebra on a single spin state $|\eta\rangle$:

$$\{\alpha_j, \alpha_k\}|\eta\rangle = 2\delta_{jk}|\eta\rangle\,, \quad \{\alpha_j, \beta\}|\eta\rangle = 0\,, \quad \beta|\eta\rangle = |\eta\rangle$$

d+1 states suffice: $\alpha_j = |0\rangle\langle j| + |j\rangle\langle 0|\,,\ \beta = 2|0\rangle\langle 0| - I\,,\ |\eta\rangle = |0\rangle$

Adiabatic algorithms for spatial search

Adiabatic algorithm (Farhi, Goldstone, Gutmann, Sipser 00):
Start in the ground state of a simple Hamiltonian and slowly change the Hamiltonian so that the ground state encodes the solution to the problem.

Simple algorithm: $H = \gamma L - |w\rangle\langle w|$
Slowly lower γ from a large value to 0.
With an appropriate schedule, can search in time
$O(N^{1/2})$ for $d>4$
$O(N^{1/2} \log^{3/2} N)$ for $d=4$

Dirac algorithm: $H = \omega \sum_{j=1}^{d} \alpha_j P_j + \gamma\beta L - \beta|w\rangle\langle w|$

Starting state $|s\rangle$ is in the middle of the spectrum, and states in middle of spectrum with ω,γ small have very little overlap on $|w\rangle$.

Open questions

What is the actual complexity of spatial search in $d=2$?

How does the algorithm work when there are multiple marked items? With non-periodic boundary conditions? Starting from a localized state?

Can this algorithm be implemented in a feasible experiment (e.g. in optical lattices)?

Other algorithms using quantum walks?