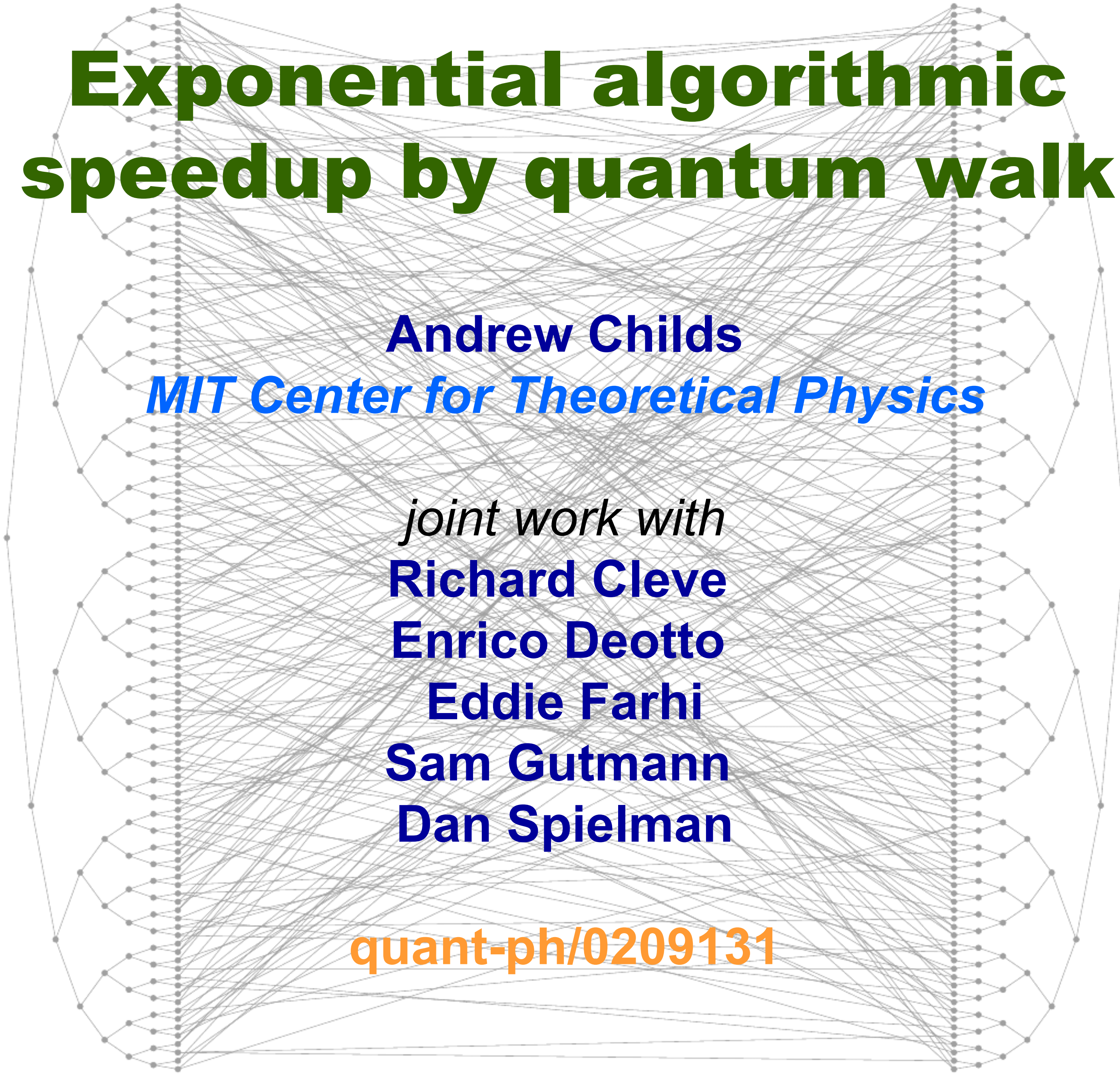

Exponential algorithmic speedup by quantum walk

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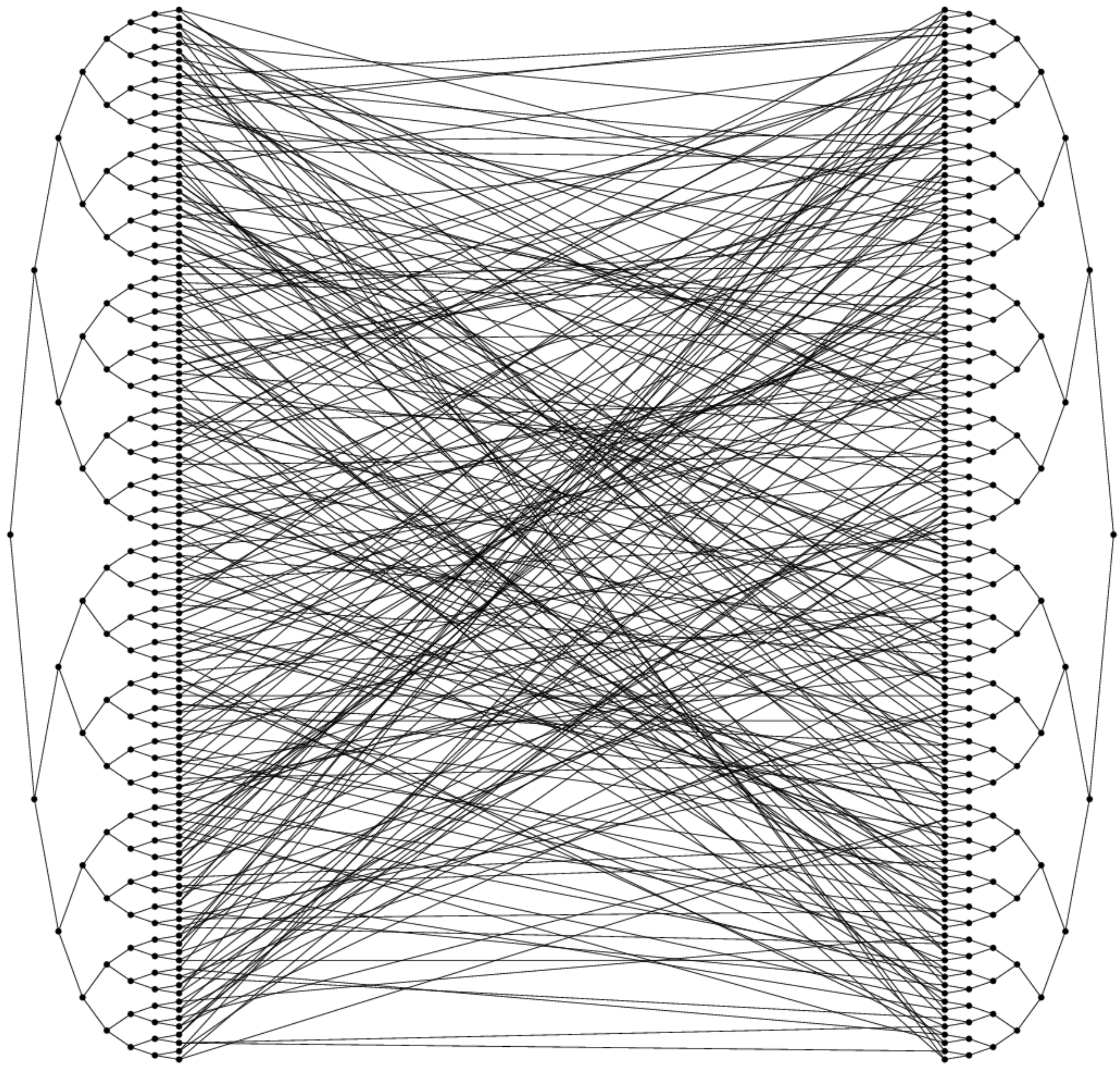
quant-ph/0209131



Motivation

- Find new kinds of quantum algorithms

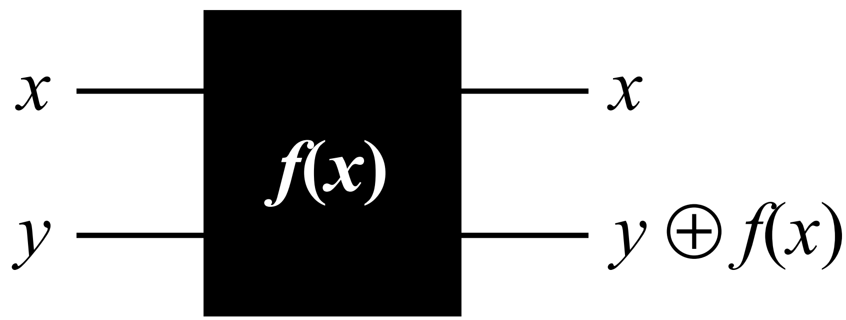
- Classical random walks → quantum walks
- Construct an (oracular) problem that is naturally suited to quantum walks
- Show that the problem can be solved efficiently using quantum walks
  - Walk finds the solution fast
  - Walk can be implemented
- Show that the problem cannot be solved efficiently using a classical computer

Black box computation

- **Standard computation:** compute a function of some data

  Example: Factoring.
  Input: An integer j
  Output: Integers k, l such that $j = kl$

- **Black box computation** (oracular computation)
  Input: A black box for a function $f(x)$
  Output: Some property of $f(x)$



Running time: count queries to $f(x)$.
Easier to obtain bounds.

Black box computation

(Physical perspective)

- **Physics experiment**

  Input: Apparatus with Hamiltonian H

  Output: Parameters c_1, c_2, …

$$|\psi_{\text{in}}\rangle \;\text{———}\; \boxed{H} \;\text{———}\; |\psi_{\text{out}}\rangle = e^{-i\,H\,t}\,|\psi_{\text{in}}\rangle$$

Determine c_1, c_2, … as fast as possible.

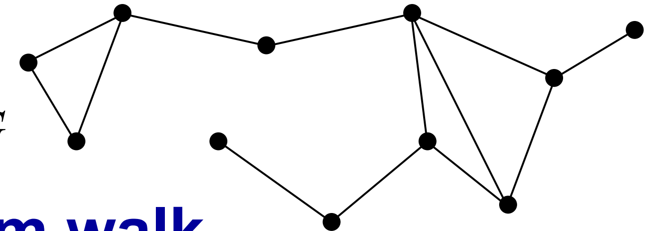
History of quantum algorithms

quantum Fourier transform

Deutsch 85	One quantum query vs. two classical queries
Deutsch/Josza 92	Exact quantum solution exponentially faster than exact classical solution
Bernstein/Vazirani 93	Superpolynomial quantum-classical separation
Simon 94	Exponential quantum-classical separation
Shor 94	Factoring/discrete log
Kitaev 95	Abelian hidden subgroup problem
van Dam/Hallgren 00	Quadratic character problems
Watrous 01	Algorithms for solvable groups
Hallgren 02	Pell's equation
Grover 96	Quadratic speedup of search
CCDFGS 02	Quantum walks

Quantum walk

graph G



Classical random walk

Differential equation

$$\frac{\mathrm{d}p_a(t)}{\mathrm{d}t} = \sum_{a'} K_{aa'}\, p_{a'}(t)$$

Generator

$$K_{aa'} = \begin{cases} \gamma & a \neq a',\ aa' \in G \\ 0 & a \neq a',\ aa' \notin G \\ -d(a)\gamma & a = a'\,. \end{cases}$$

Probability conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_a p_a(t) = 0$$

Quantum walk

Differential equation (Schrödinger)

$$i\frac{\mathrm{d}}{\mathrm{d}t}\langle a|\psi(t)\rangle = \sum_{a'} \langle a|H|a'\rangle\langle a'|\psi(t)\rangle$$

Generator

$$\cancel{\langle a|H|a'\rangle = K_{aa'}}$$

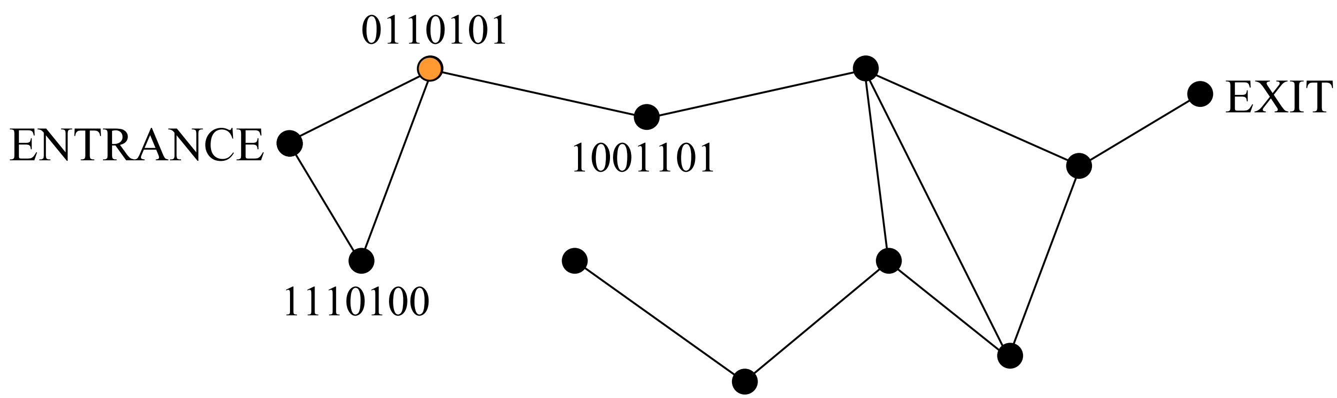
$$\langle a|H|a'\rangle = \begin{cases} \gamma & a \neq a',\ aa' \in G \\ 0 & \text{otherwise.} \end{cases}$$

Probability conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_a |\langle a|\psi(t)\rangle|^2 = 0$$

Another choice: Quantum analogue of discrete random walk.
Hilbert space cannot be just the vertices [Meyer, J. Stat. Phys. **85**, 511 (1996)] so one must introduce a "quantum coin."

Black box graph traversal problem



Names of vertices: random $2n$-bit strings ($n = \lceil \log N \rceil$)

Name of ENTRANCE is known

Oracle outputs the names of adjacent vertices

$v_c(a)$ = cth neighbor of a

Examples:

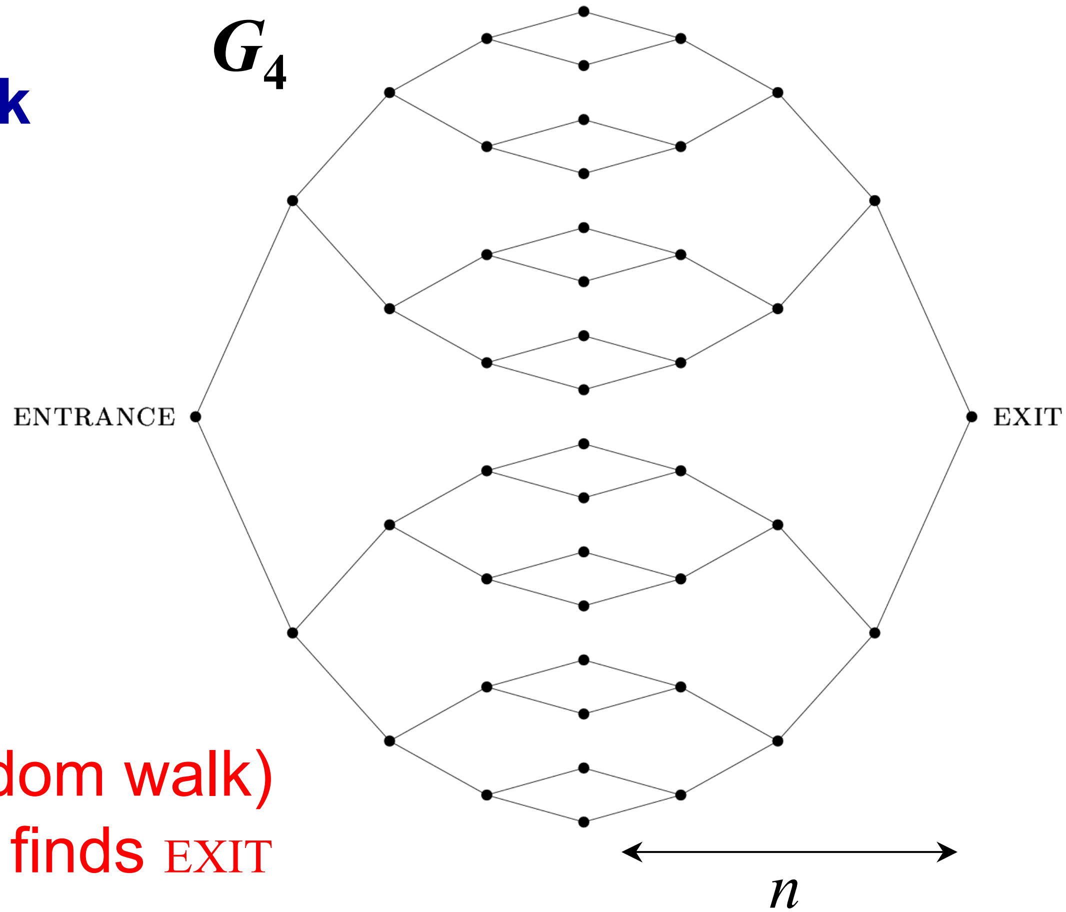
$v_1(\text{ENTRANCE}) = 0110101$	$v_1(0110101) = 1001101$
$v_2(\text{ENTRANCE}) = 1110100$	$v_2(0110101) = \text{ENTRANCE}$
$v_3(\text{ENTRANCE}) = 1111111$	$v_3(0110101) = 1110100$
$v_4(\text{ENTRANCE}) = 1111111$	$v_4(0110101) = 1111111$

Example: G_n

Classical random walk
Time to reach EXIT is exponential in n.

Quantum walk
Time to reach EXIT is linear in n.

But there is a (non-random walk) classical algorithm that finds EXIT in polynomial time!

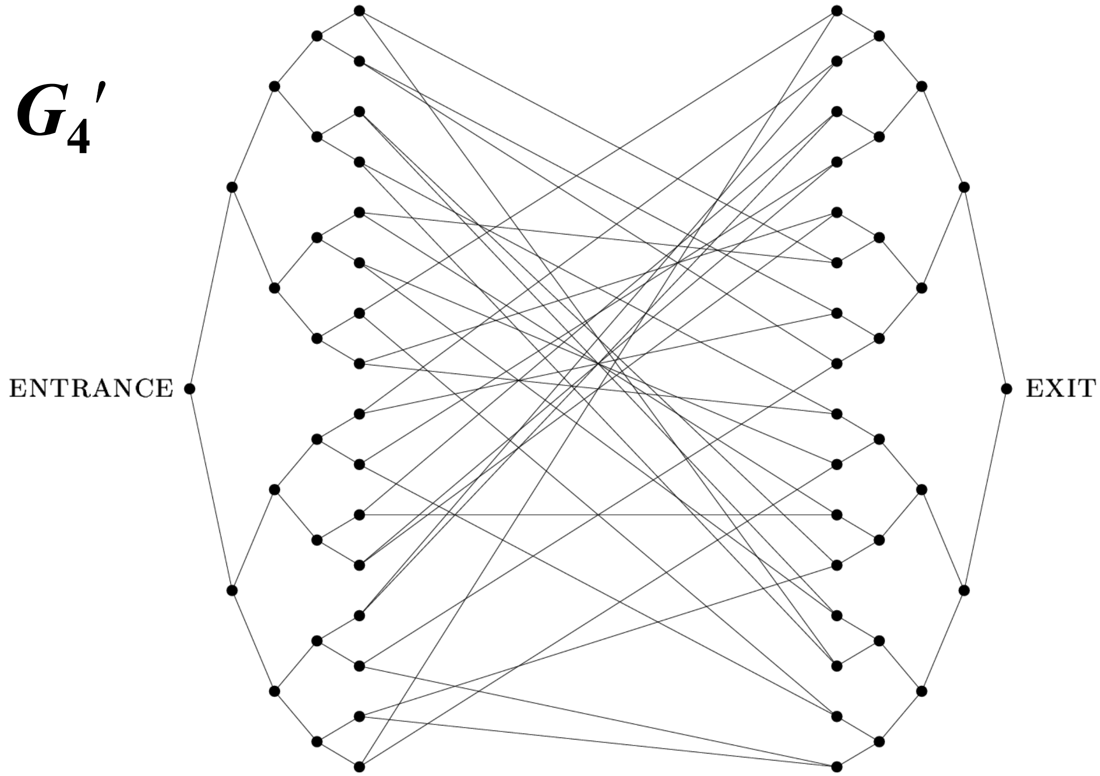


$O(2^n)$ vertices

Childs, Farhi, Gutmann, QIP **1**, 35 (2002).

A harder graph: G_n'

G_4'



(Actually a distribution on graphs)

Connection: a random cycle that alternates sides

Reduction of $G_n{}'$ to a line

Column subspace

$$|\,\mathrm{col}\, j\rangle = \frac{1}{\sqrt{N_j}} \sum_{a \in \text{column } j} |a\rangle$$

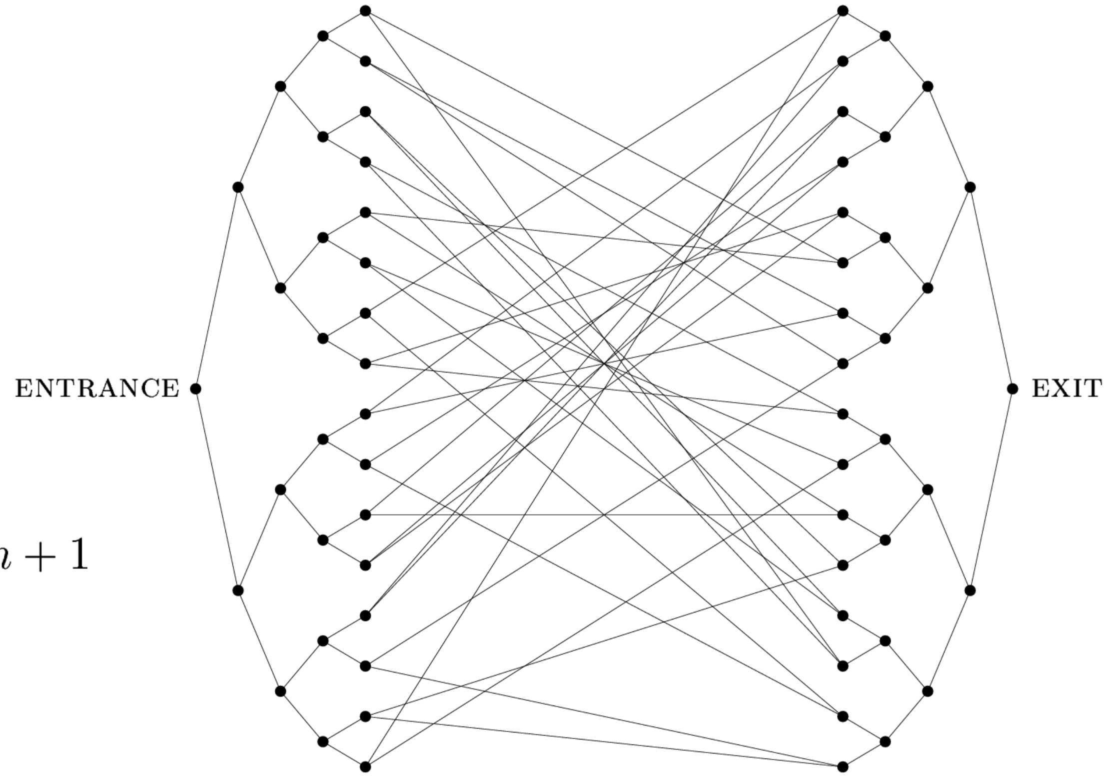
where

$$N_j = \begin{cases} 2^j & 0 \le j \le n \\ 2^{2n+1-j} & n+1 \le j \le 2n+1 \end{cases}$$

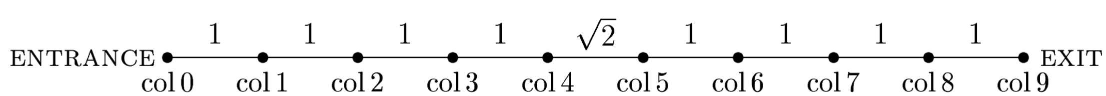
Reduced Hamiltonian

$$\langle \mathrm{col}\, j | H | \mathrm{col}(j+1)\rangle$$

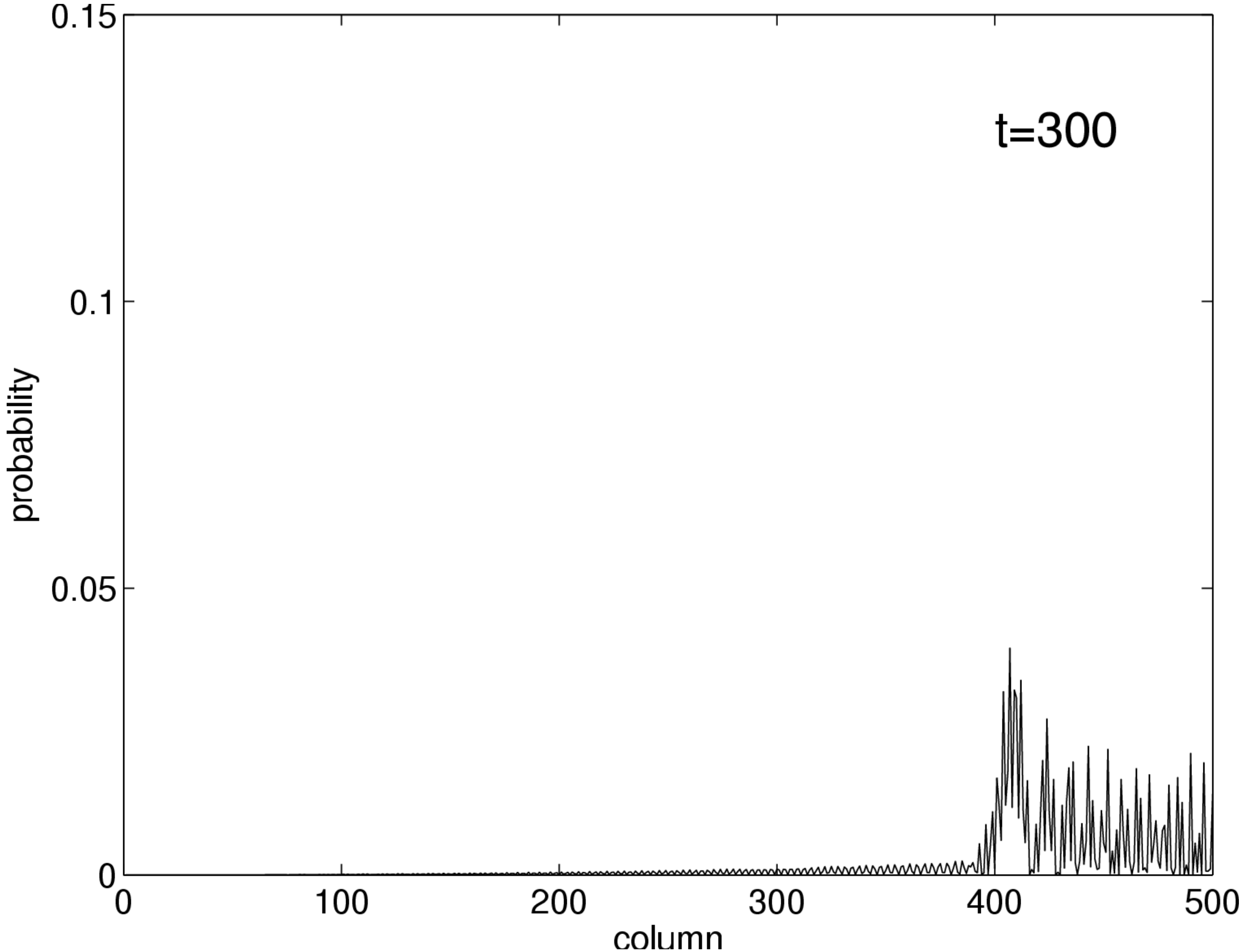
$$= \begin{cases} \sqrt{2}\gamma & 0 \le j \le n-1\,, \\ & n+1 \le j \le 2n \\ 2\gamma & j = n \end{cases}$$



$$\gamma = \frac{1}{\sqrt{2}}$$

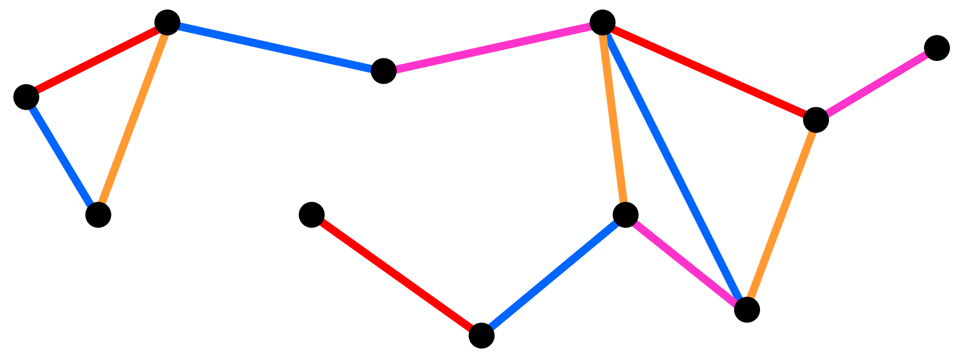


Quantum walk on G'_{250}



Implementing the quantum walk 1

General edge-colored graph



$v_c(a)$ = neighbor of a connected by an edge of color c

$v_c(v_c(a)) = a$ for $v_c(a) \in G$

Hilbert space: states $|a, b, r\rangle$

$2n$ bits $\quad 2n$ bits $\quad 1$ bit

Vertex states: $|a, 0, 0\rangle$
$a \in G$

Using the oracle, can compute

$$V_c|a,b,r\rangle = |a, b \oplus v_c(a), r \oplus f_c(a)\rangle \qquad f_c(a) = \begin{cases} 0 & v_c(a) \in G \\ 1 & v_c(a) \notin G \end{cases}$$

Implementing the quantum walk 2

Tools for simulating Hamiltonians

- Linear combination $e^{-i(H_1+\cdots+H_k)t} = \left(e^{-iH_1 t/j} \cdots e^{-iH_k t/j}\right)^j + O(k\|[H_p, H_q]\|t^2/j)$
- Unitary conjugation $U e^{-iHt} U^\dagger = e^{-iUHU^\dagger t}$

A simple Hamiltonian

$$\begin{aligned} T|a,b,0\rangle &= |b,a,0\rangle \\ T|a,b,1\rangle &= 0 \end{aligned}$$

Graph Hamiltonian

$$H = \sum_c V_c^\dagger T V_c$$

Proof:

$$\begin{aligned} H|a,0,0\rangle &= \sum_c V_c T|a, v_c(a), f_c(a)\rangle \\ &= \sum_c \delta_{0,f_c(a)} V_c |v_c(a), a, 0\rangle \\ &= \sum_{c:\, v_c(a)\in G} |v_c(a), a \oplus v_c(v_c(a)), f_c(v_c(a))\rangle \\ &= \sum_{c:\, v_c(a)\in G} |v_c(a), 0, 0\rangle \end{aligned}$$

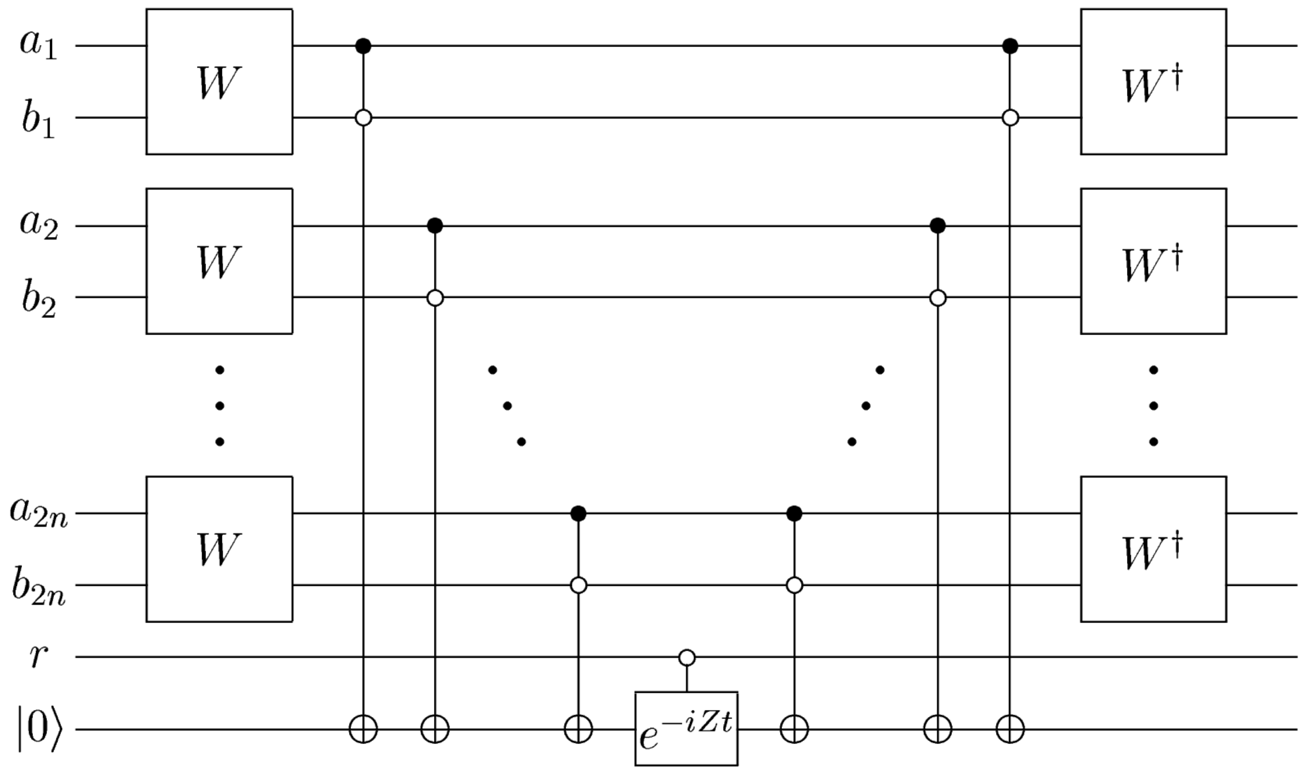
Simulating *T*

Simulate
$$\begin{aligned} T|a,b,0\rangle &= |b,a,0\rangle \\ T|a,b,1\rangle &= 0 \end{aligned}$$

i.e.,
$$T = \left(\bigotimes_{l=1}^{2n} S^{(l,2n+l)}\right) \otimes |0\rangle\langle 0|$$

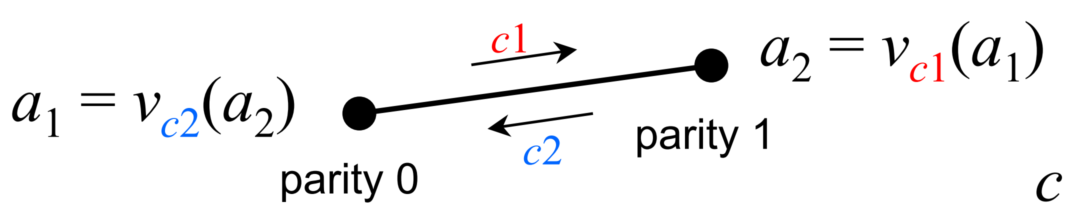
SWAP: $S|z_1 z_2\rangle = |z_2 z_1\rangle$

Diagonalize:
$$\begin{aligned} W|00\rangle &= |00\rangle \\ W\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) &= |01\rangle \\ W\tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) &= |10\rangle \\ W|11\rangle &= |11\rangle \end{aligned}$$



Coloring G_n'

- General oracle: $v_c(v_c(a)) \neq a$
- Construct an oracle that has this property
- G_n' is bipartite. Define a parity bit:
  0 if vertex is in an even column
  1 if vertex is in an odd column
- Parity of ENTRANCE is 0, and we can easily modify the oracle to keep track of parity
- Color of an edge depends on parity:
  – Parity 0 $\quad c = (c_{\text{in}}, c_{\text{out}})$
  – Parity 1 $\quad c = (c_{\text{out}}, c_{\text{in}})$



$$c = (c2, c1)$$

Quantum algorithm

- Start in the state $|\textsc{entrance}, 0, 0\rangle$
- Simulate the quantum walk for a time $t = \mathrm{poly}(n)$
- Measure in the computational basis
- If $\textsc{exit}$ is found, stop; otherwise repeat

- **Theorem:** If t is chosen uniformly in $[0, n^4]$ then the probability of finding $\textsc{exit}$ is greater than $1/4n$.

$\Rightarrow$ Quantum walk algorithm finds the $\textsc{exit}$ with high probability using a polynomial number of gates and oracle queries.

Classical lower bound 1

Consider the set of vertices visited by the algorithm.

Restrict to a connected subgraph

- Number of vertices in G_n': $O(2^n)$
- Number of possible names: 2^{2n}
- Probability of guessing a valid name at random: $O(2^{-n})$

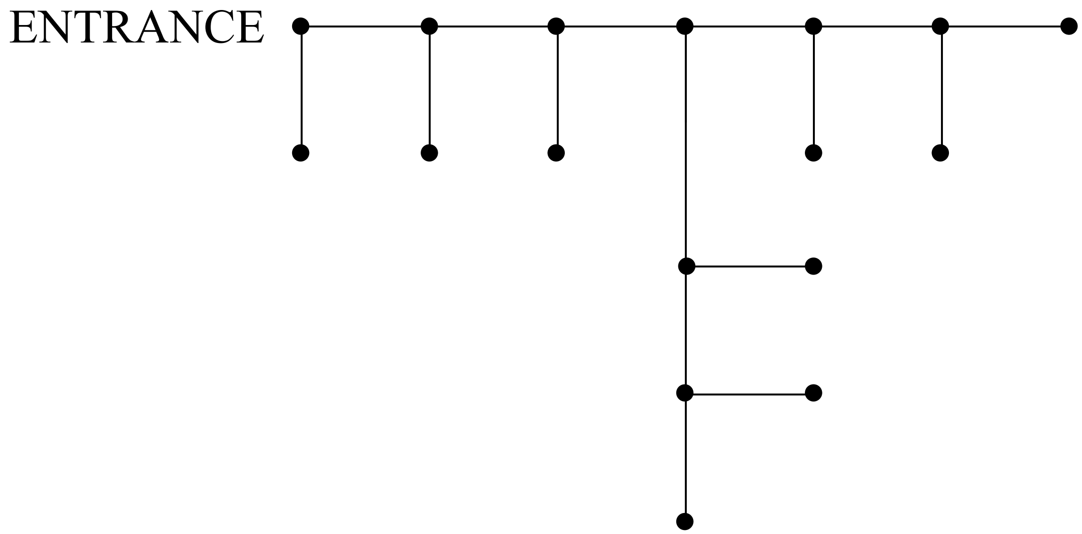
Restrict to a subtree

- Allow the algorithm to win if it finds the EXIT *or* if it finds a cycle

Classical lower bound 2

Restrict to random embeddings of rooted binary trees

- If the algorithm does not find a cycle, then it is simply tracing out a rooted subtree of G_n'



- Consider an arbitrary rooted binary tree with t vertices. What is the probability that a random embedding into G_n' produces a cycle or finds the EXIT?

Classical lower bound 3

- Answer: if $t < 2^{n/6}$, then the probability is less than $3 \cdot 2^{-n/6}$.

- Putting it all together, we have

  Theorem: Any classical algorithm that makes at most $2^{n/6}$ queries to the oracle finds the EXIT with probability at most $4 \cdot 2^{-n/6}$.

Remarks

- Provably exponential quantum-classical separation using quantum walks
- Q: Why does the algorithm work?

  A: Quantum interference!
- Find the EXIT without finding a path from ENTRANCE to EXIT
- Could put the coloring in the classical lower bound
- Easy to formulate as a decision problem

Open problems

- Is it possible to implement the walk for a general graph with no restriction on the initial state?

- Are there *interesting* computational problems that can be solved using quantum walks?