From optimal measurement to efficient quantum algorithms for the hidden subgroup problem and beyond

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- Simulating quantum dynamics
- Factoring
- Discrete log
- Pell's equation
- Abelian HSP
- Some nonabelian HSPs
- Estimating gauss sums
- Legendre symbol/polynomial reconstruction
- Graph traversal
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Techniques

- Fourier sampling
- Quantum walk
- Adiabatic optimization
- Trace estimation
- Optimal measurement

Outline

- The hidden subgroup problem (HSP)
- Optimal measurements for distinguishing quantum states
- Dihedral HSP
- Heisenberg HSP
- Unlabeled hidden shift problem
- Summary and open problems

The hidden subgroup problem

Problem: Fix a group G (known) and a subgroup H (unknown).
Given a black box that computes $f\colon G \to S$ that is

- Constant on any particular left coset of H in G
- Distinct on different left cosets of H in G

(We say that f hides H.)

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Even for very simple groups (e.g., $G = \mathbb{Z}_2^n$), a classical algorithm provably requires exponentially many queries of f to find H.

Most interesting cases of the HSP

- Abelian groups
  Applications to factoring, discrete log, Pell's equation, etc.
  Can be solved efficiently

- Dihedral group
  Applications to lattice problems [Regev 2002]
  Subexponential-time algorithm [Kuperberg 2003]

- Symmetric group
  Application to graph isomorphism
  No nontrivial algorithms

Efficient algorithms for the HSP

- Abelian groups [Shor 1994; Boneh, Lipton 1995; Kitaev 1995]
- Normal subgroups [Hallgren, Russell, Ta-Shma 2000]
- "Almost abelian" groups [Grigni, Schulman, Vazirani^2 2001]
- "Near-Hamiltonian" groups [Gavinsky 2004]
- $(\mathbb{Z}_2^n \times \mathbb{Z}_2^n) \rtimes \mathbb{Z}_2$ [Püschel, Rötteler, Beth 1998]
- $\mathbb{Z}_{p^k}^n \rtimes \mathbb{Z}_2$, smoothly solvable groups [Friedl, Ivanyos, Magniez, Santha, Sen 2002]
- p-hedral: $\mathbb{Z}_N \rtimes \mathbb{Z}_p$, $p{=}\phi(N)/\mathrm{poly}(\log N)$ prime, N prime [Moore, Rockmore, Russell, Schulman 2004]
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- ➡ $\mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ [Inui, Le Gall 2004]
- ➡ $\mathbb{Z}_p^r \rtimes \mathbb{Z}_p$, r constant (including Heisenberg, $r{=}2$) NEW!

Standard approach to the HSP

Compute uniform superposition of function values:

$$\frac{1}{\sqrt{|G|}}\sum_{g\in G}|g\rangle \mapsto \frac{1}{\sqrt{|G|}}\sum_{g\in G}|g, f(g)\rangle$$

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Now we can (without loss of generality) perform a Fourier transform over G, and measure which irreducible representation the state is in (weak Fourier sampling).

Distinguishing quantum states

Problem: Given a quantum state ρ chosen from an ensemble of states ρ_i with a priori probabilities p_i, determine i.

This can only be done perfectly if the states are orthogonal. In general, we would just like a high probability of success: maximize $\Sigma_i \; p_i \, \mathrm{tr}(\rho_i E_i)$.

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Bad news: For some groups, it is necessary to make joint measurements on $\Omega(\log|G|)$ copies. [Moore, Russell, Schulman 2005-6; Hallgren, Rötteler, Sen 2006]

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Can we use this as a principle to find quantum algorithms?

Optimal measurement

Theorem. [Holevo 1973, Yuen-Kennedy-Lax 1975]
Given an ensemble of quantum states ρ_i with a priori probabilities p_i, the measurement with POVM elements E_i maximizes the probability of successfully identifying the state if and only if $R = R^\dagger$ and $R \geq p_i \rho_i$ for all i, where

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In general, it is nontrivial to find a POVM that satisfies these conditions (although it is a semidefinite program!).

But for all the cases discussed in this talk, the optimal measurement is a particularly simple POVM, the *pretty good measurement*.

Pretty good measurement

Given states ρ_i with a priori probabilities p_i, define POVM elements

$$E_i := p_i \frac{1}{\sqrt{\Sigma}} \rho_i \frac{1}{\sqrt{\Sigma}} \qquad \text{where} \qquad \Sigma := \sum_i p_i \rho_i$$

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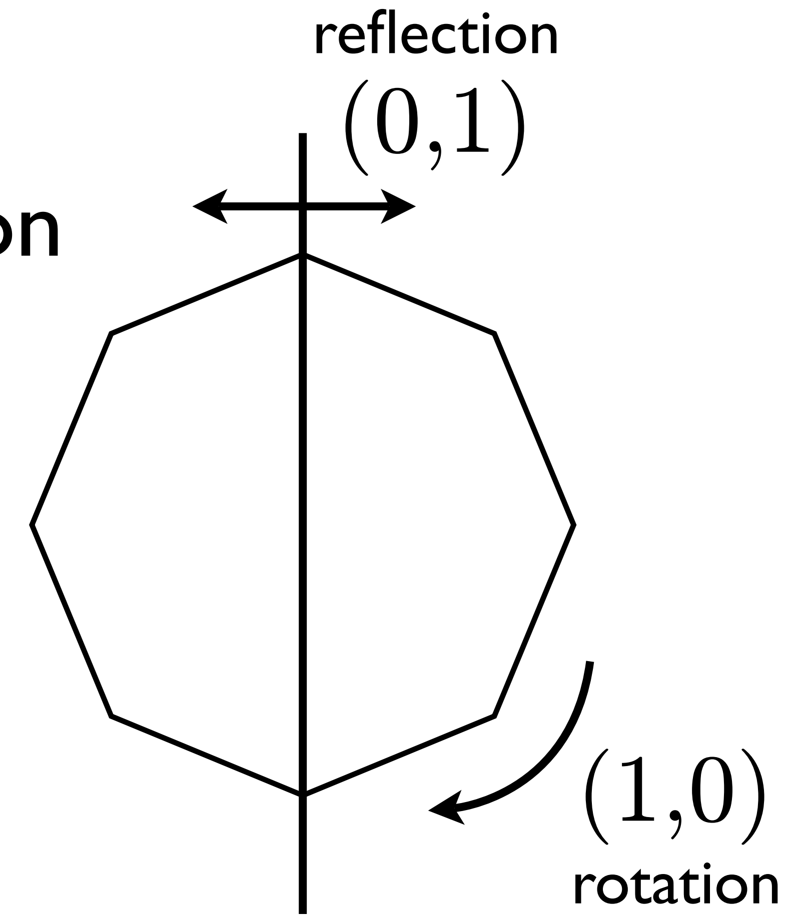
The PGM often does a pretty good job of distinguishing the ρ_i.
In fact, sometimes it is optimal! (Check Holevo/YKL conditions)

Dihedral group $(\mathbb{Z}_N \rtimes \mathbb{Z}_2)$

Symmetry group of an N-sided regular polygon



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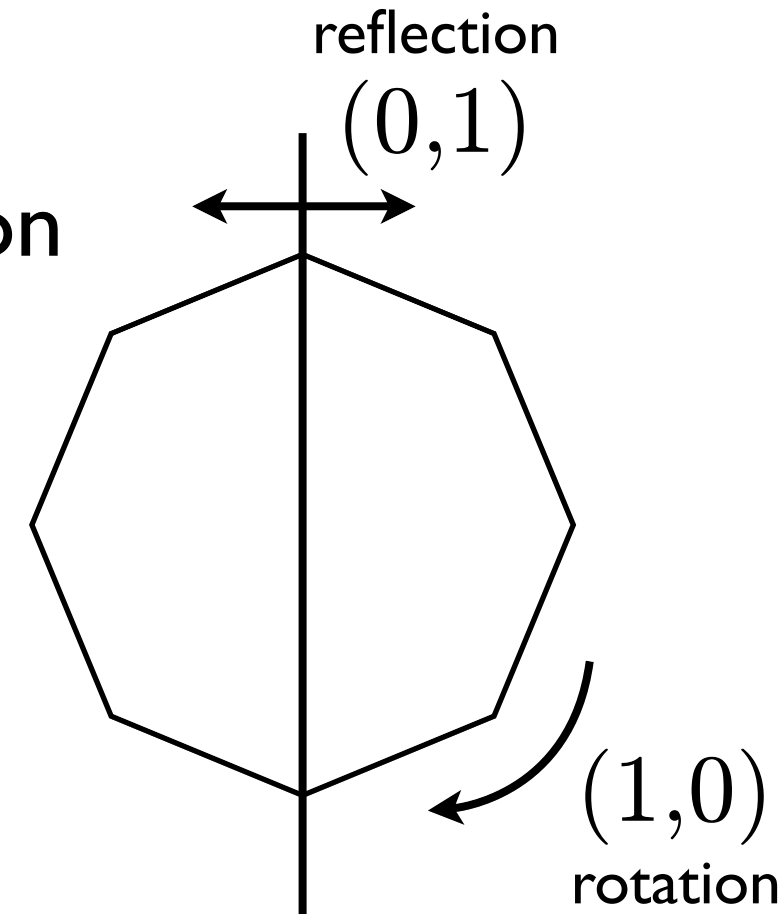
$$(a,b)(c,d) = (a + (-1)^b c, b + d)$$



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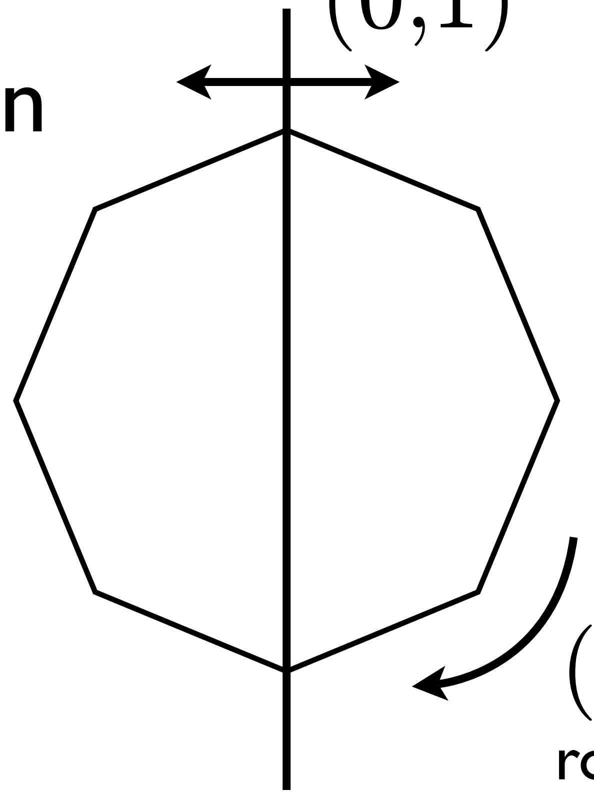

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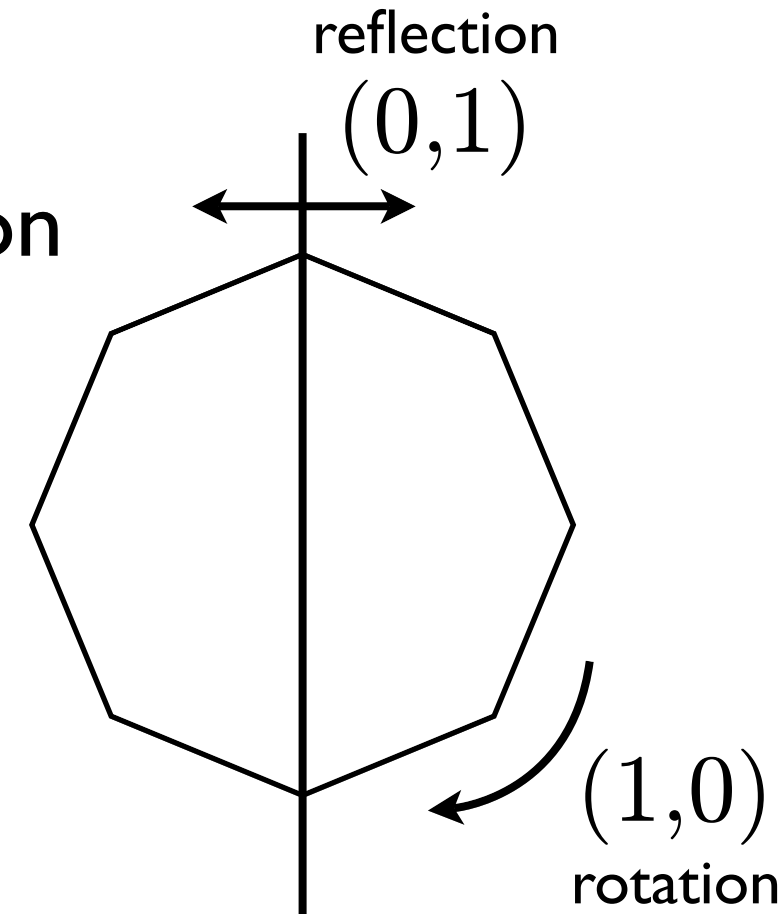

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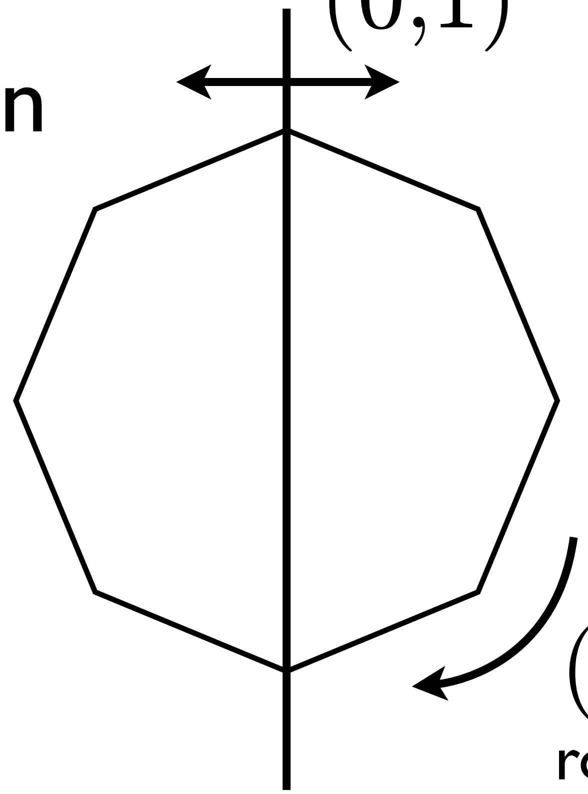

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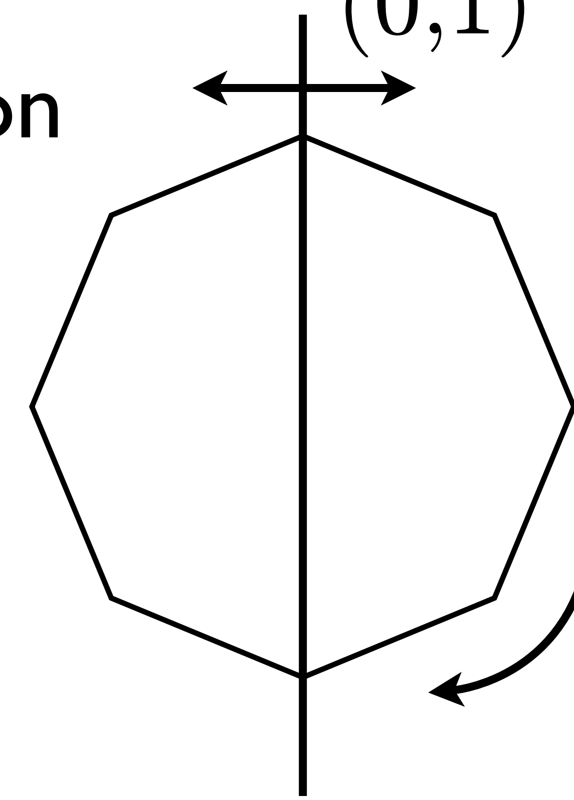

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By symmetry, we can measure x wlog (Fourier sampling: measure which irreducible representation)

Multiple dihedral coset states

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$$= \frac{1}{\sqrt{2^k}} \sum_{w \in \mathbb{Z}_N} \omega^{wa} \sqrt{\eta_w^{\vec{x}}} |S_w^{\vec{x}}\rangle$$

solutions of *subset sum problem*:

$$S_w^{\vec{x}} := \{\vec{b} \in \mathbb{Z}_2^k : \vec{b}\cdot\vec{x} = w\}$$

$$\eta_w^{\vec{x}} := |S_w^{\vec{x}}|$$

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Subset sum and DHSP

The PGM (which is optimal) can be implemented unitarily by doing the inverse of the *quantum sampling* transformation:

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Questions:

- How big must k be so that the solutions of the subset sum problem are nearly uniformly distributed?
- For such values of k, can we quantum sample from the subset sum solutions?

Subset sum problem

Problem: Given k integers $x_1,\ldots,x_k$ from $\mathbb{Z}_N$ and a target w from $\mathbb{Z}_N$, find a subset of the k integers that sum to the target (i.e., find $b_1,\ldots,b_k$ from $\mathbb{Z}_2$ so that $b \cdot x = w$).

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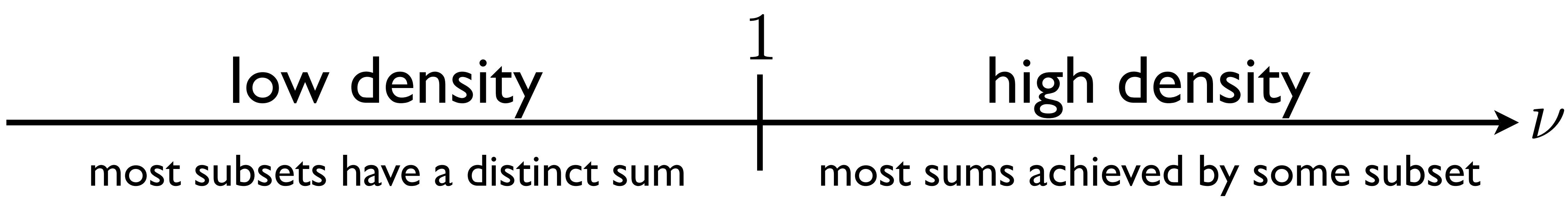

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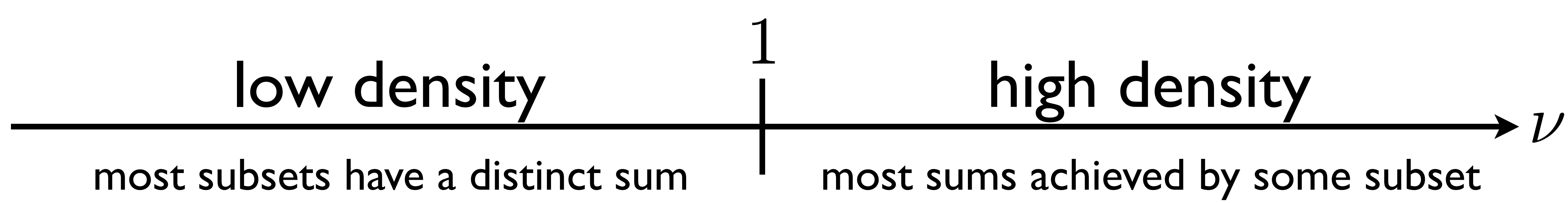

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most subsets have a distinct sum

most sums achieved by some subset

subset sum $\Rightarrow$ DHSP [Regev 2002]

PGM succeeds

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$\mathrm{poly}(k)$ quantum DHSP algorithm

[Kuperberg 2003]

General approach

- Cast problem as a state distinguishability problem (e.g., coset states for HSP)
- Express the states in terms of an average-case algebraic problem (e.g., subset sum for dihedral HSP)
- Perform the pretty good measurement on k copies of the states:
  - Choose k large enough that the measurement succeeds with reasonably high probability (this happens if the average-case problem typically has many solutions)
  - Implement the measurement by solving the problem on average (quantum sampling from the set of solutions)

The Heisenberg group

Subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ a & c & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}$

Semidirect product $\mathbb{Z}_p^2 \rtimes_\varphi \mathbb{Z}_p$

$$\varphi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_p^2) \quad \text{with} \quad \varphi(c)(a,b) = (a + bc, b)$$

$$(a,b,c)(a',b',c') = (a + a' + b'c, b + b', c + c')$$

Group of $p \times p$ unitary matrices

$\langle X, Z \rangle = \{\omega^a X^b Z^c : a, b, c \in \mathbb{Z}_p\}$ where

$$X := \sum_{x \in \mathbb{Z}_p} |x+1\rangle\langle x| , \quad Z := \sum_{x \in \mathbb{Z}_p} \omega^x |x\rangle\langle x| , \quad \omega := e^{2\pi \mathrm{i}/p}$$

Heisenberg subgroups

Fact: To solve the HSP in the Heisenberg group, it is sufficient to distinguish the order p subgroups $\langle(a,b,1)\rangle = \{(a,b,1)^j : j \in \mathbb{Z}_p\}$

$$(a,b,1)^2 = (a,b,1)(a,b,1) = (2a+b, 2b, 2)$$
$$(a,b,1)^3 = (a,b,1)(2a+b, 2b, 2) = (3a+3b, 3b, 3)$$
$$(a,b,1)^4 = (a,b,1)(3a+2b, 3b, 3) = (4a+6b, 4b, 4)$$
$$\vdots$$
$$(a,b,1)^j = (ja + \tbinom{j}{2} b, jb, j)$$

Average-case problem: Two quadratic equations in k variables.

Heisenberg HSP algorithm

Two copies of the coset states are sufficient to distinguish these subgroups. The optimal measurement can be implemented by solving a pair of quadratic equations in two variables.

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This algorithm implements an entangled measurement across r coset states. This is encouraging, since entangled measurements are information-theoretically necessary for some groups!*

*But not for the Heisenberg group [Radhakrishnan, Rötteler, Sen 2005], although no *efficient* single-register algorithm is known for this case.

Generalized abelian hidden shift problem

Problem: Given a function $f : \{0, 1, \ldots, M-1\} \times \mathbb{Z}_N \to S$ satisfying $f(b, x) = f(b+1, x+s)$ for $b = 0, 1, \ldots, M-2$, find the value of the *hidden shift* $s \in \mathbb{Z}_N$.

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This is an instance of integer programming in k dimensions. Lenstra's algorithm (based on LLL lattice basis reduction) solves this efficiently for k constant. $k{=}\log N / \log M \Rightarrow$ efficient algorithm for any $M{=}N^\epsilon$ for fixed $\epsilon{>}0$.

Original problem	k	Average-case problem	Solution
Abelian HSP	1	Linear equations	Easy
Metacyclic HSP $\mathbb{Z}_N \rtimes \mathbb{Z}_p,\ p = \phi(N)/\operatorname{poly}(\log N)$	1	Discrete log	Shor's algorithm
$\mathbb{Z}_p^r \rtimes \mathbb{Z}_p$ (r=2 is Heisenberg)	r	Polynomial equations	Buchburger's algorithm, elimination
Generalized abelian hidden shift problem, $M{=}N^\epsilon$	$1/\epsilon$	Integer programming	Lenstra's algorithm
Dihedral HSP	$\log N$	Subset sum	?
Symmetric group HSP	$n \log n$	?	?

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- Are there other hidden subgroup problems (besides dihedral & symmetric groups) with practical applications?