Teleportation-based approaches to universal quantum computation with single-qubit measurements

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joint work with

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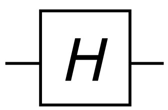
Resource requirements for quantum computation

- Well-defined Hilbert space with tensor product structure.
- Prepare the system in a standard initial state.
- Perform a universal set of state transformations.
- Read out the result of the computation.
- Isolate the computer from the effects of its environment.

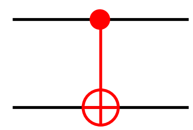
These are abstract requirements. What physical operations must be performed?

Circuit model of quantum computation

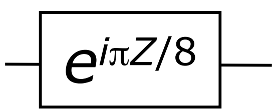
- Prepare a simple initial state $|00\cdots0\rangle$.
- Perform a universal set of 1- and 2-qubit unitary gates (e.g., H, CNOT, exp[$i\pi Z/8$]).



$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$= \begin{pmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}$$

- Make a measurement in the computational basis.

Measurement is universal

Two kinds of models:

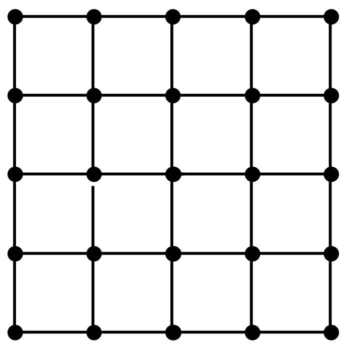
1.) Product initial state, few-qubit measurements
(Gottesman & Chuang, Nielsen, Fenner & Zhang, Leung)

- Prepare initial state $|00\cdots0\rangle$.
- Perform a sequence of 2-qubit measurements, conditioned on results of previous measurements.

2.) Entangled initial state, one-qubit measurements
(Raussendorf & Briegel)

- Prepare an entangled initial state.
- Perform a sequence of 1-qubit measurements, conditioned on results of previous measurements.

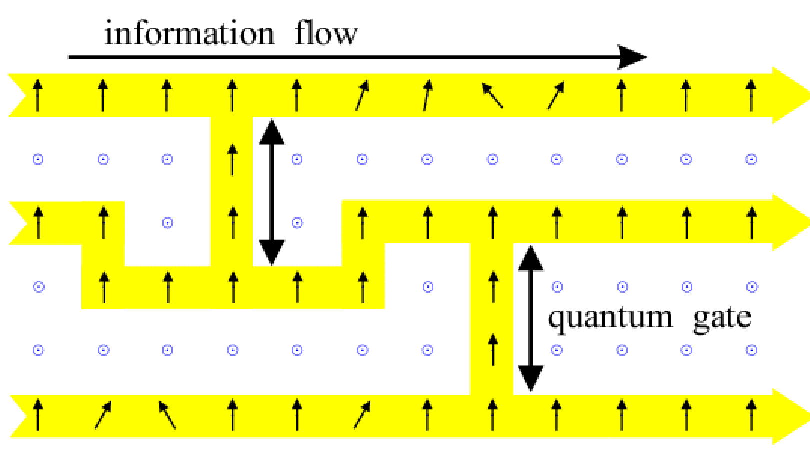
Quantum computation with cluster states



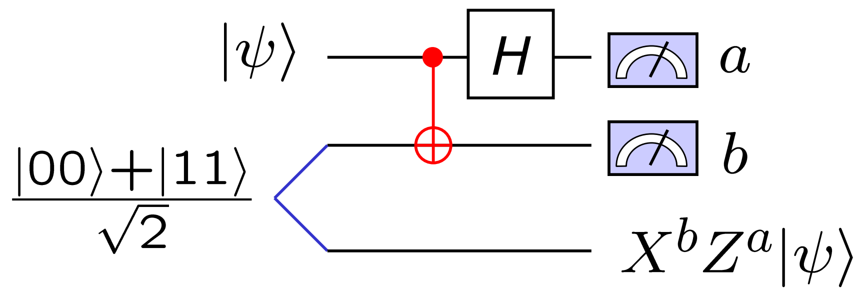
$$\exp\left[-i\frac{\pi}{4}\sum_{\langle j,k\rangle}\sigma_z^{(j)}\sigma_z^{(k)}\right]\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n}$$

Simultaneous +1 eigenstate of $\begin{matrix} & Z & \\ Z & X & Z \\ & Z & \end{matrix}$ at each vertex.

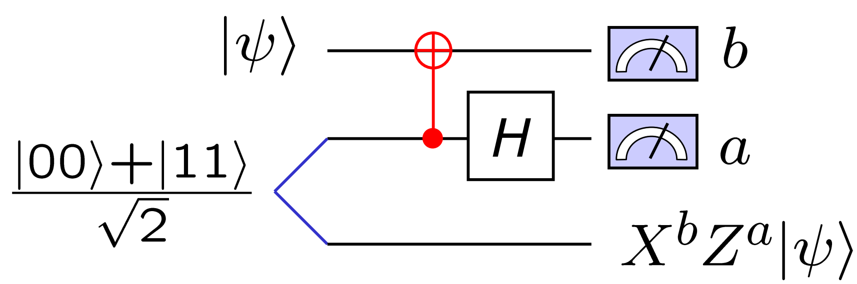
Use a sequence of measurements to manipulate quantum information:



Teleportation



or equivalently



Teleportation-based universality of single qubit measurements

Teleport logical qubits between different physical qubits.

Choice of measurement basis allows us to perform different basic operations. We will show that for certain initial states, it is possible to do a universal set of operations.

Problem: Teleportation induces Pauli errors.

Fortunately, they are known!

Maintain the state of the computation in the form $P|\psi\rangle$ where $|\psi\rangle$ is the desired state and P is a known Pauli error.

Clifford group

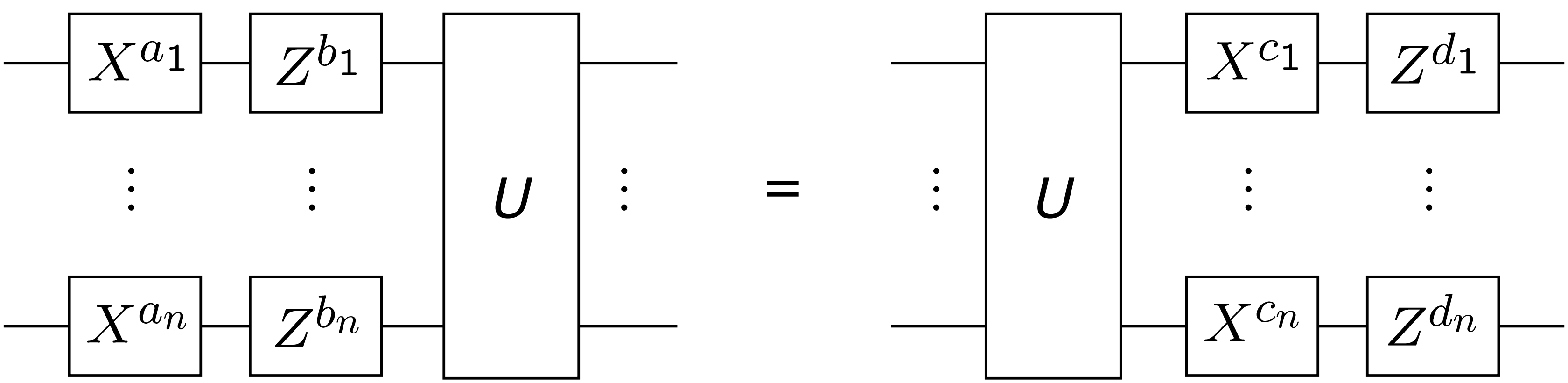
Pauli operators:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

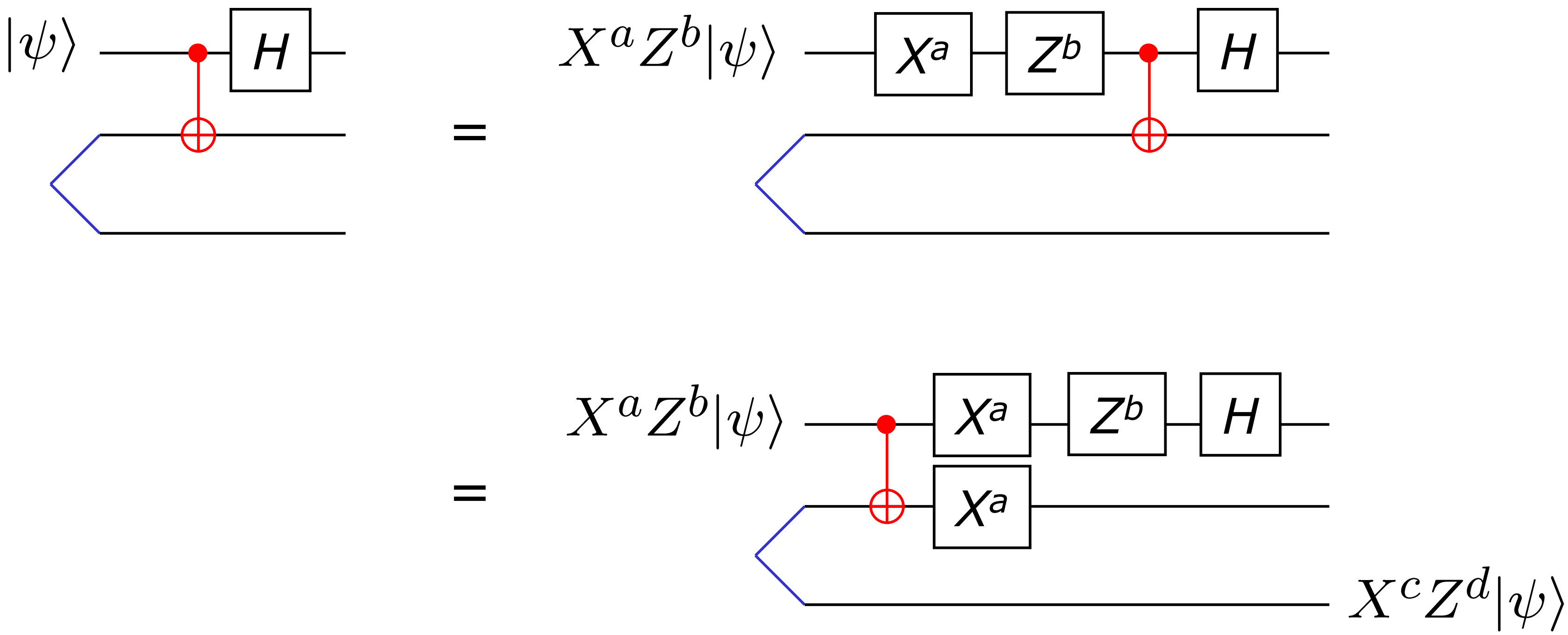
Tensor products of *n* Pauli operators form a group under multiplication, the *Pauli group* C_1.

Clifford group: $C_2 = \{U : UC_1U^\dagger \subseteq C_1\}$ (normalizer of C_1)

Operations *U* in C_2 are significant in quantum error correction because they send Pauli errors to Pauli errors.

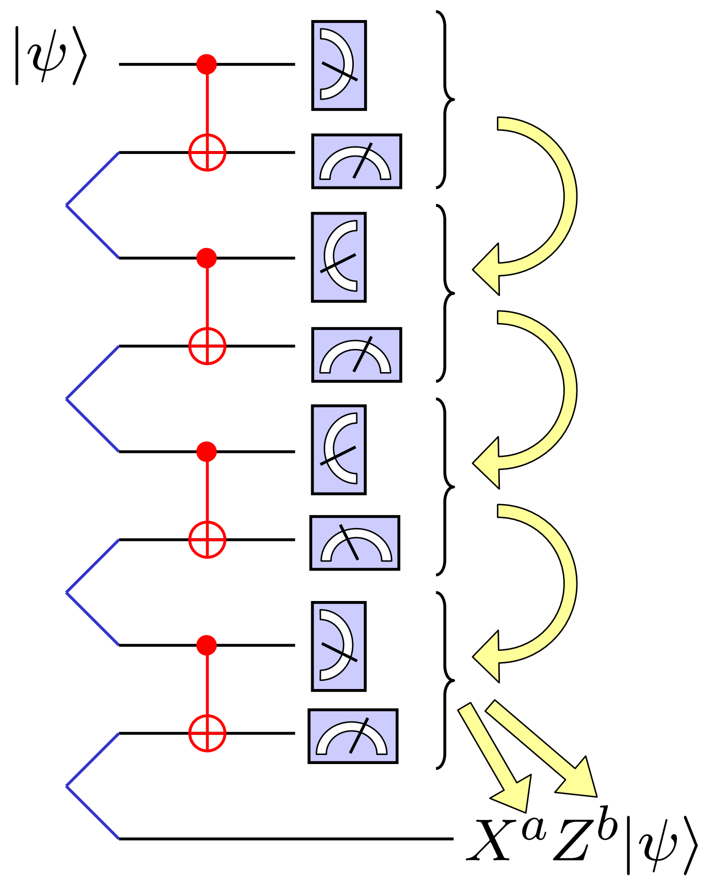


Teleportation with input errors



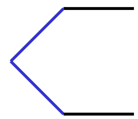
By changing the measurement basis depending on the known values of *a* and *b*, we can teleport $|\psi\rangle$ as if there were no input errors.

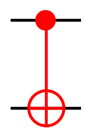
Chained teleportation

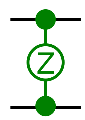


Notation

 =

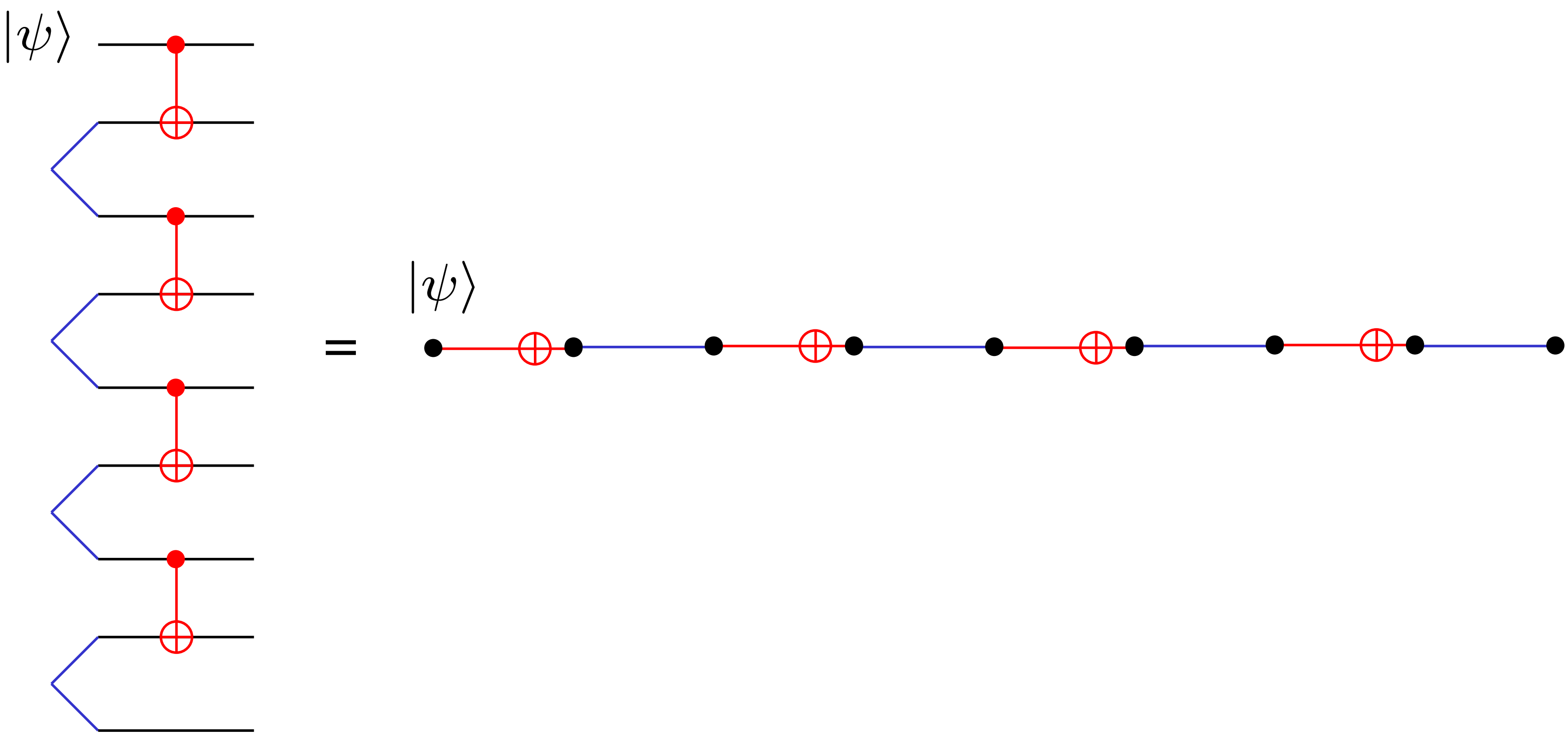
 =  $= \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

 =  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

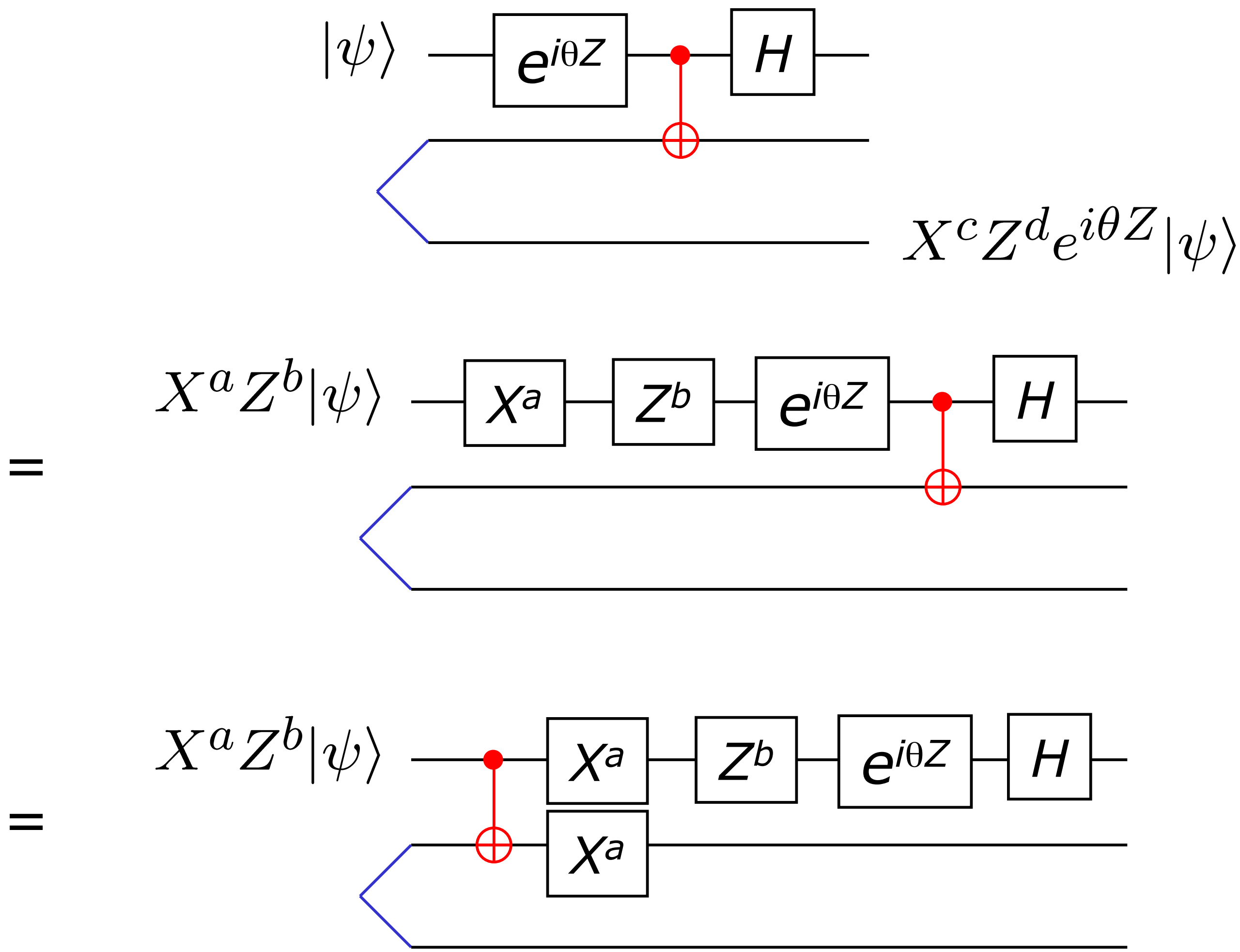
= 

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

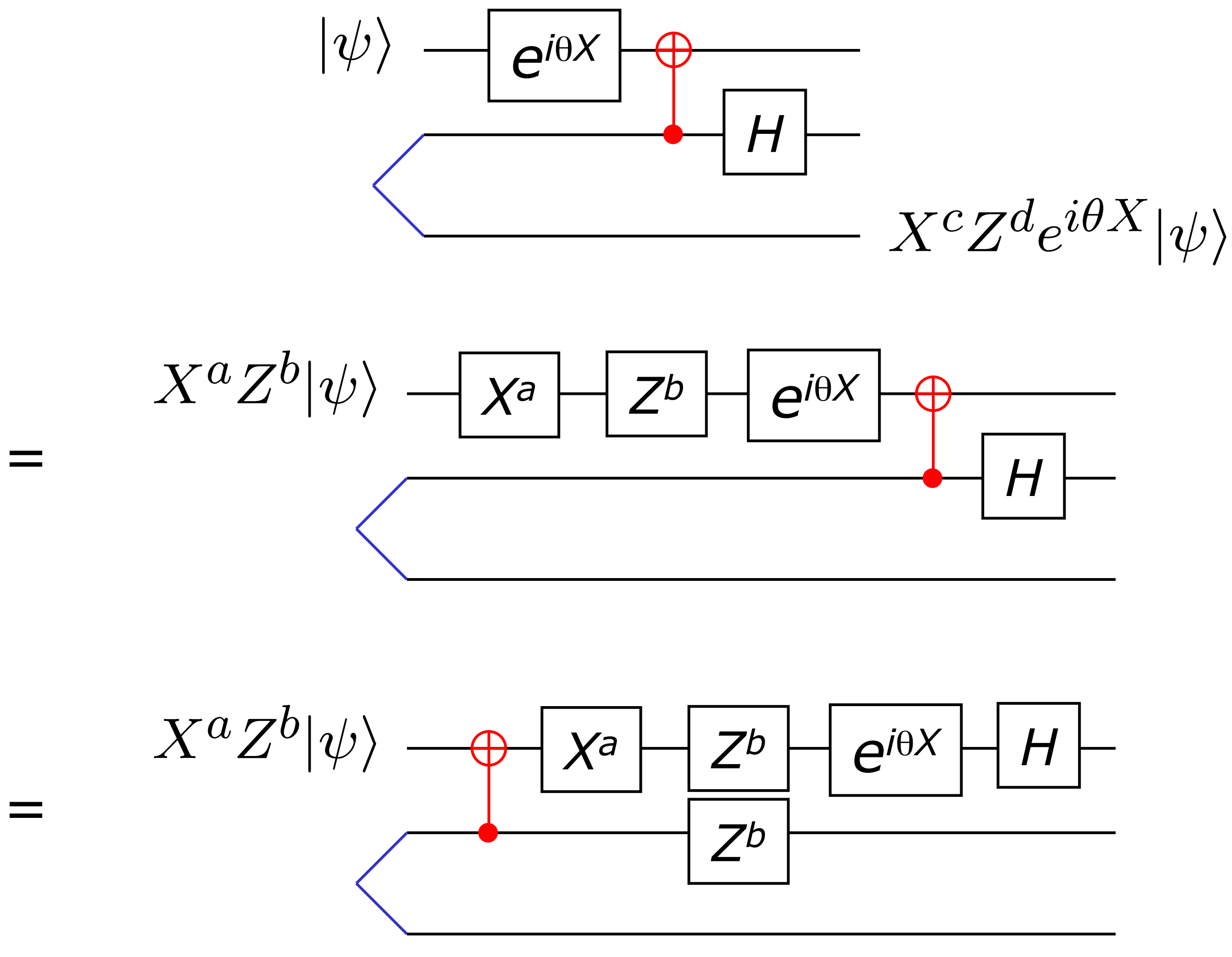
Chained teleportation



Teleporting Z rotations

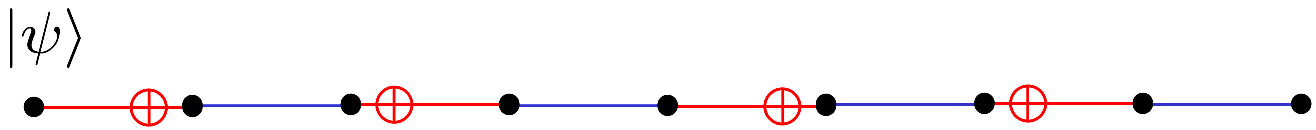


Teleporting X rotations

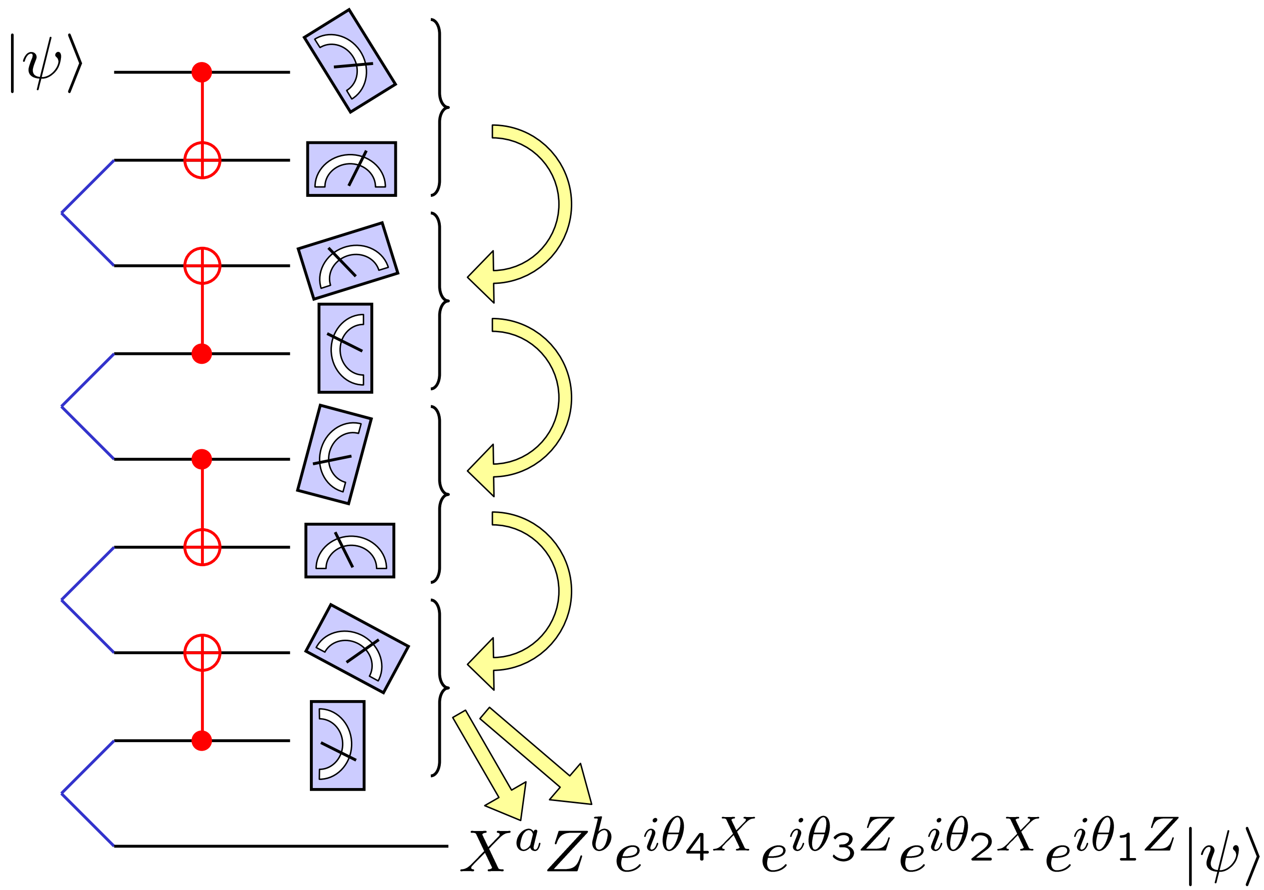


Sequence of single-qubit operations

The state



can be used to apply a sequence of single-qubit unitary transformations to $|\psi\rangle$.



Interactions between logical qubits

How can we implement interactions between different (logical) qubits?

This is easy to do with a state that depends on the particular circuit being implemented.

Instead, we want to use a fixed state. Our choice of *single-qubit measurements* will determine whether *two-qubit gates* are performed.

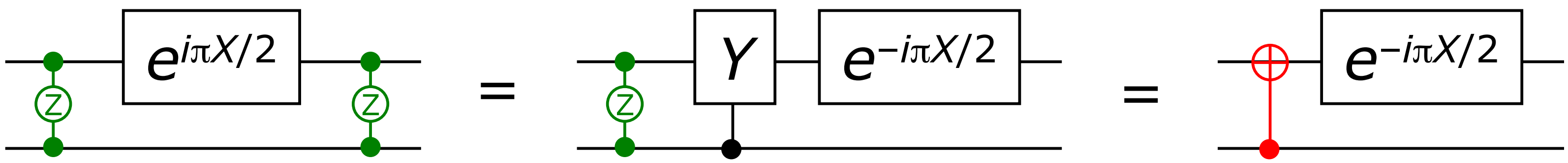
Two approaches:

1) Alternating two-qubit gates that can be made to cancel or combine.

2) Routing the logical qubits through the two-qubit gates.

Cancellation approach

but



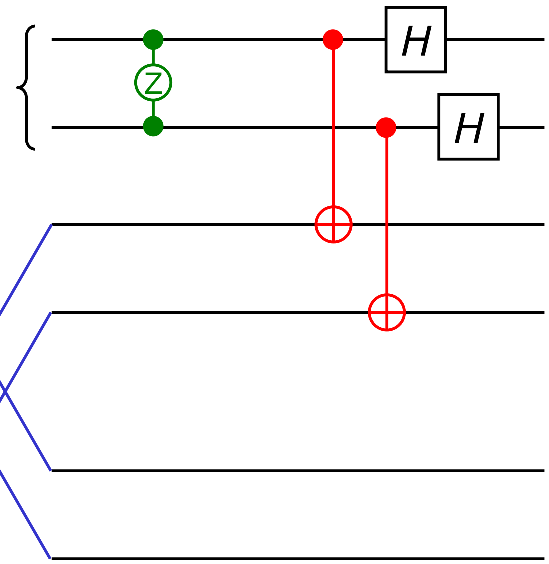
which is a universal two-qubit gate.

Thus, to implement selective two-qubit operations, it is sufficient to be able to implement nonselective controlled-phase gates (and selective one-qubit operations).

Since controlled-phase is in the Clifford group, this is easy!

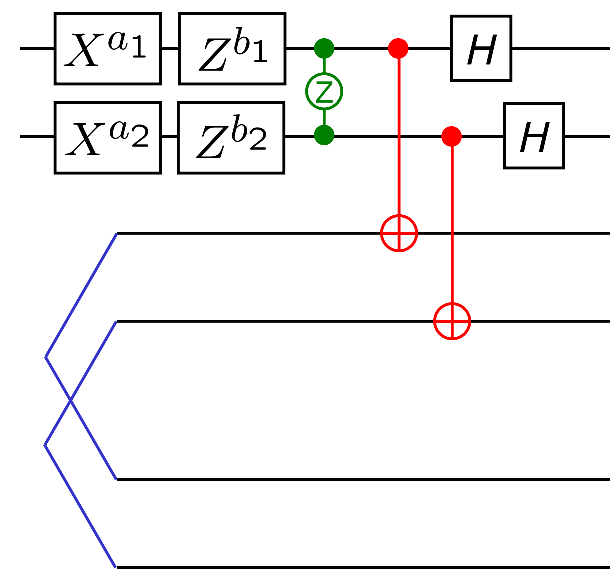
Teleporting a controlled-phase gate

$|\psi\rangle$

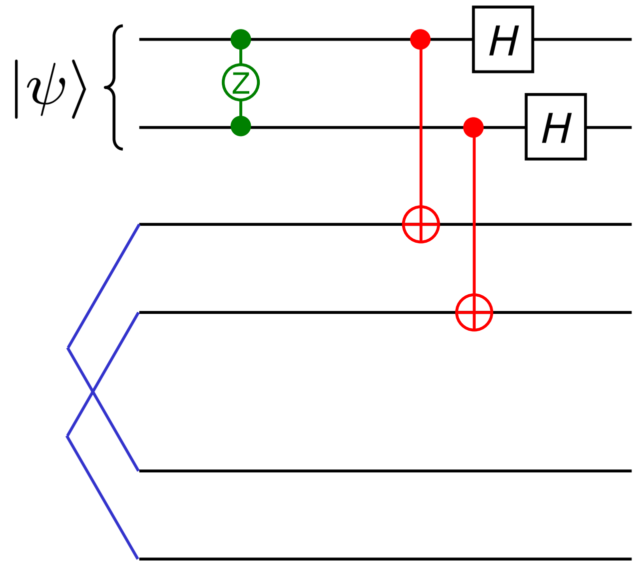


=

$(X^{a_1}Z^{b_1} \otimes X^{a_2}Z^{b_2})|\psi\rangle$

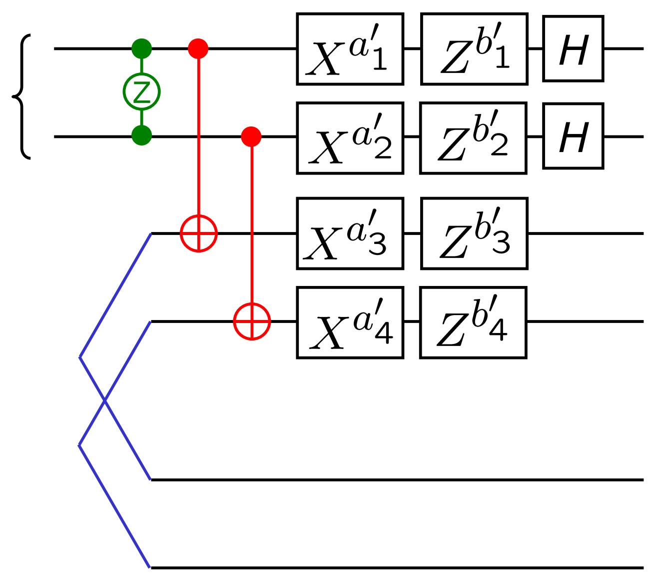


Teleportating a controlled-phase gate

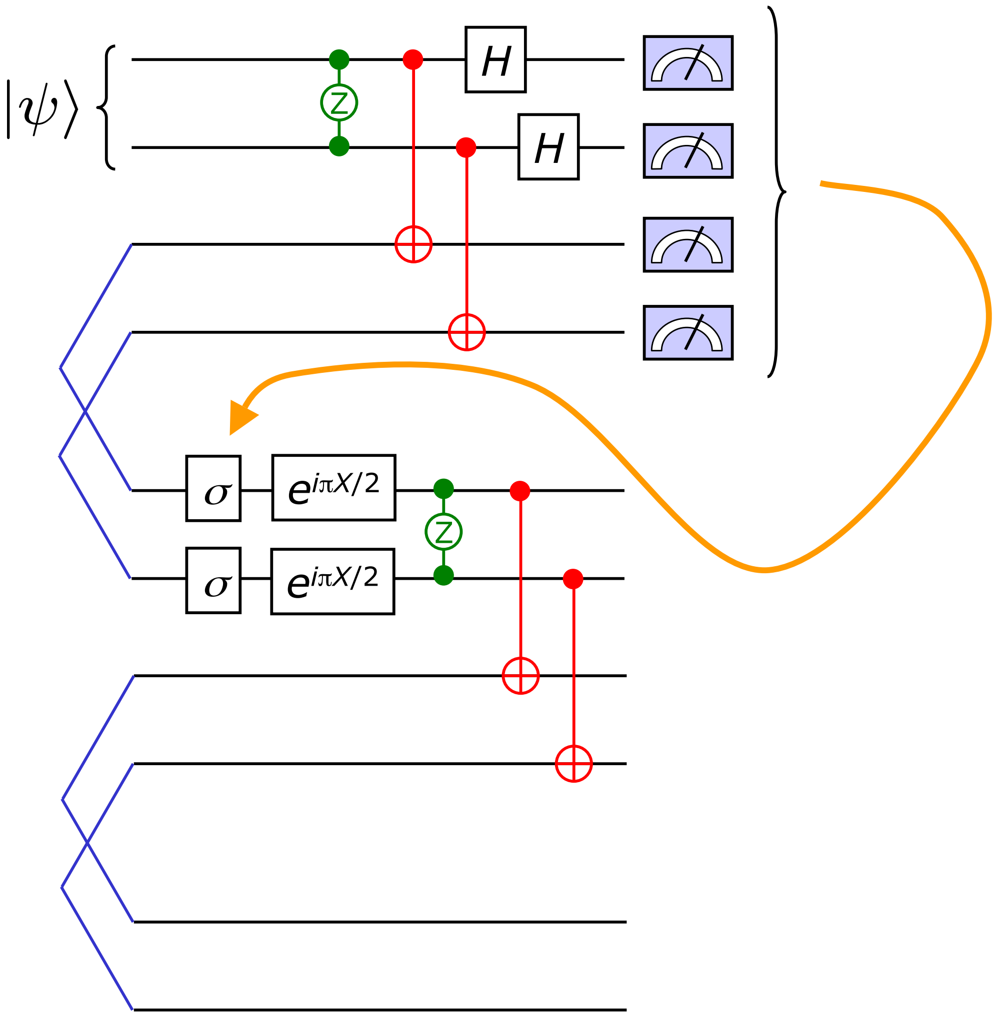


=

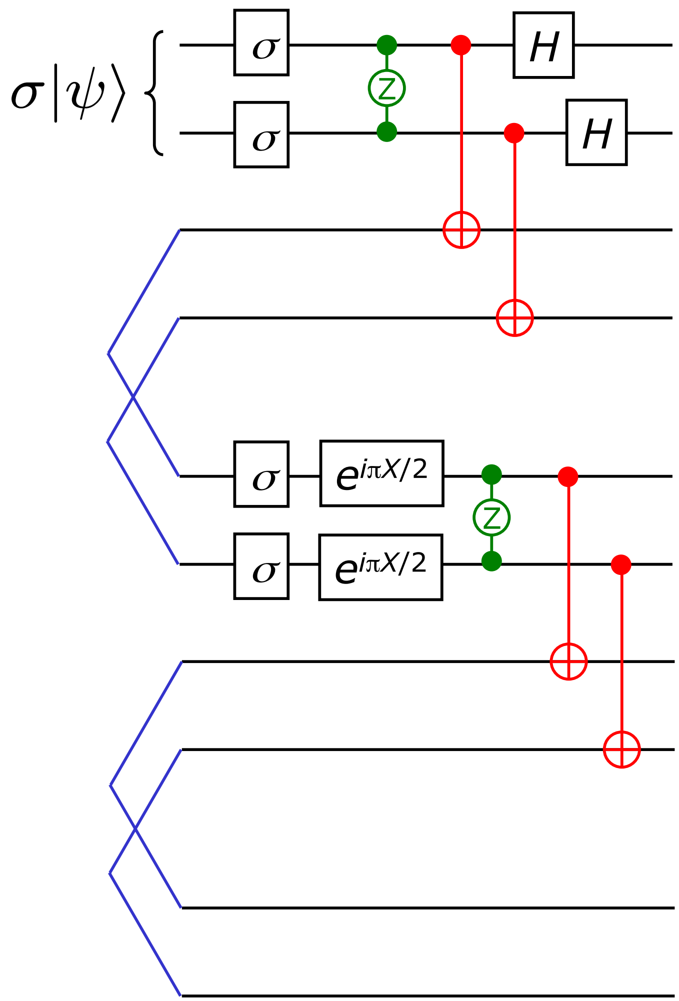
$$(X^{a_1}Z^{b_1} \otimes X^{a_2}Z^{b_2})|\psi\rangle$$



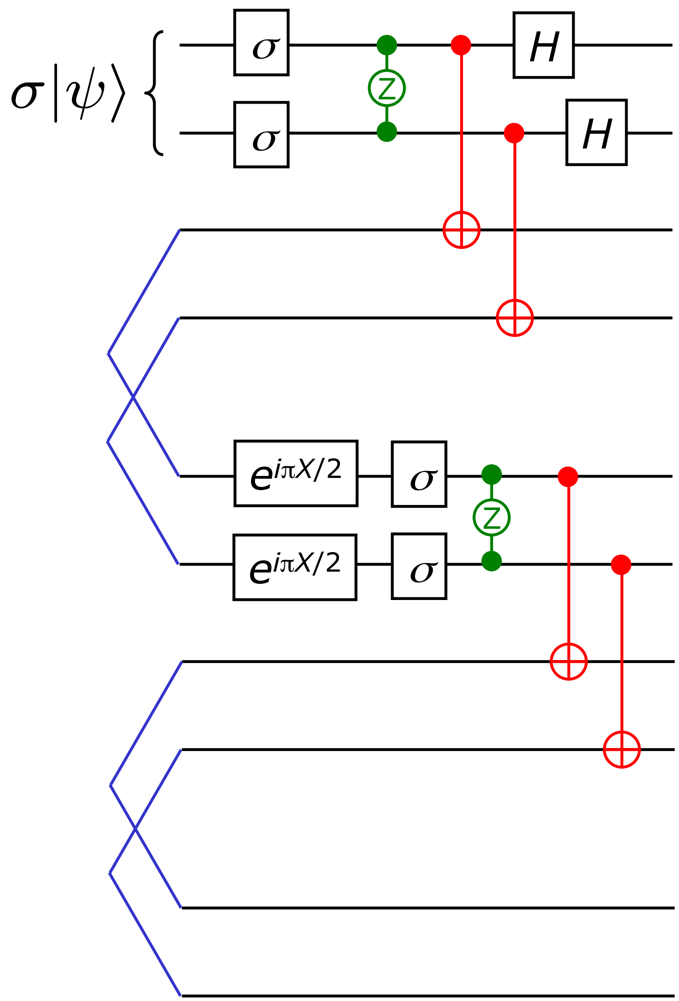
Teleportation circuit for cancellation approach



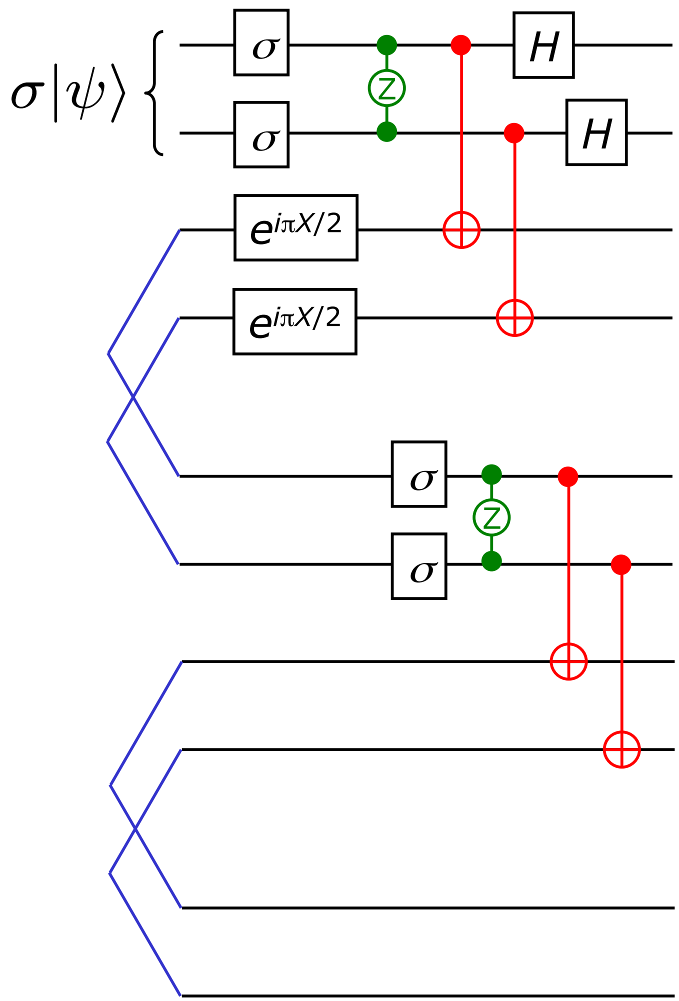
Teleportation circuit for cancellation approach



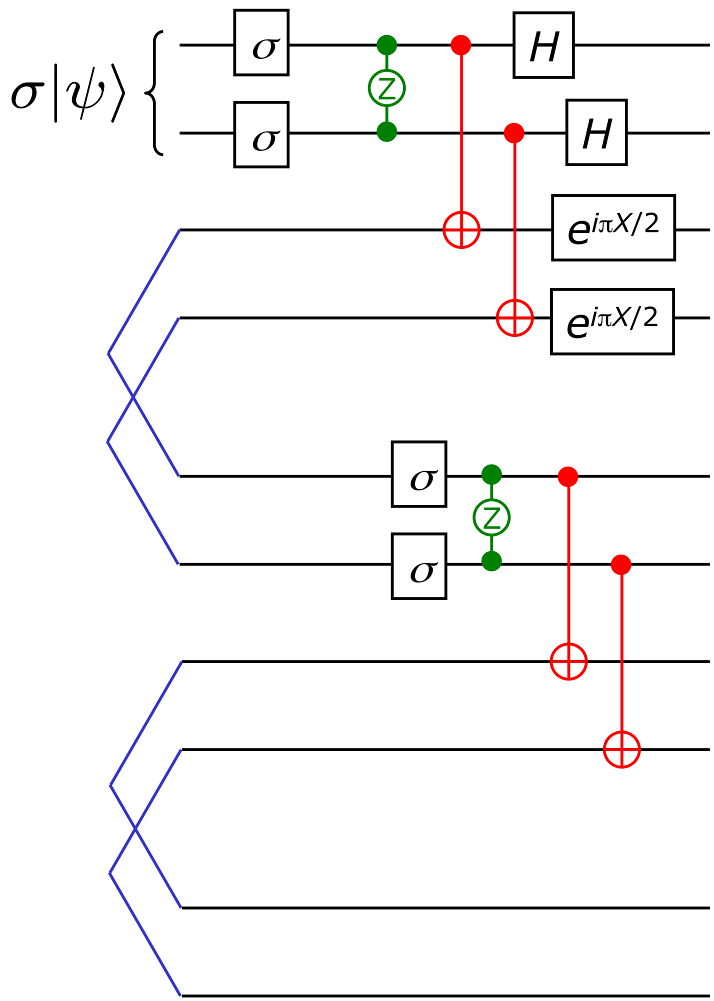
Teleportation circuit for cancellation approach



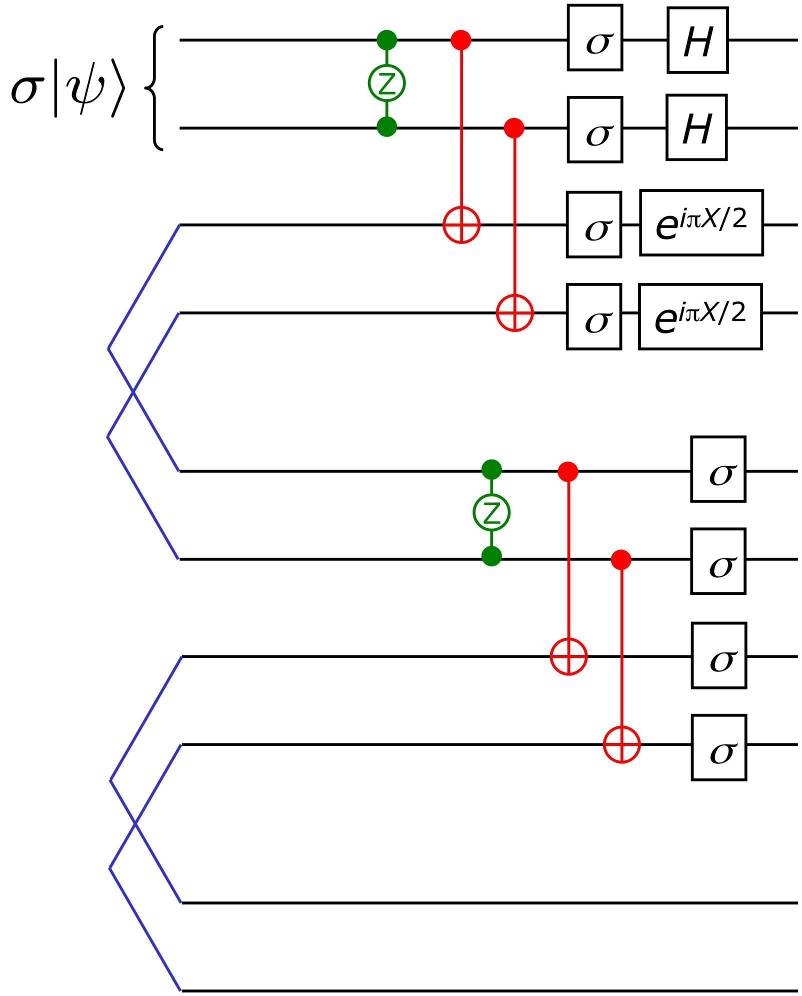
Teleportation circuit for cancellation approach



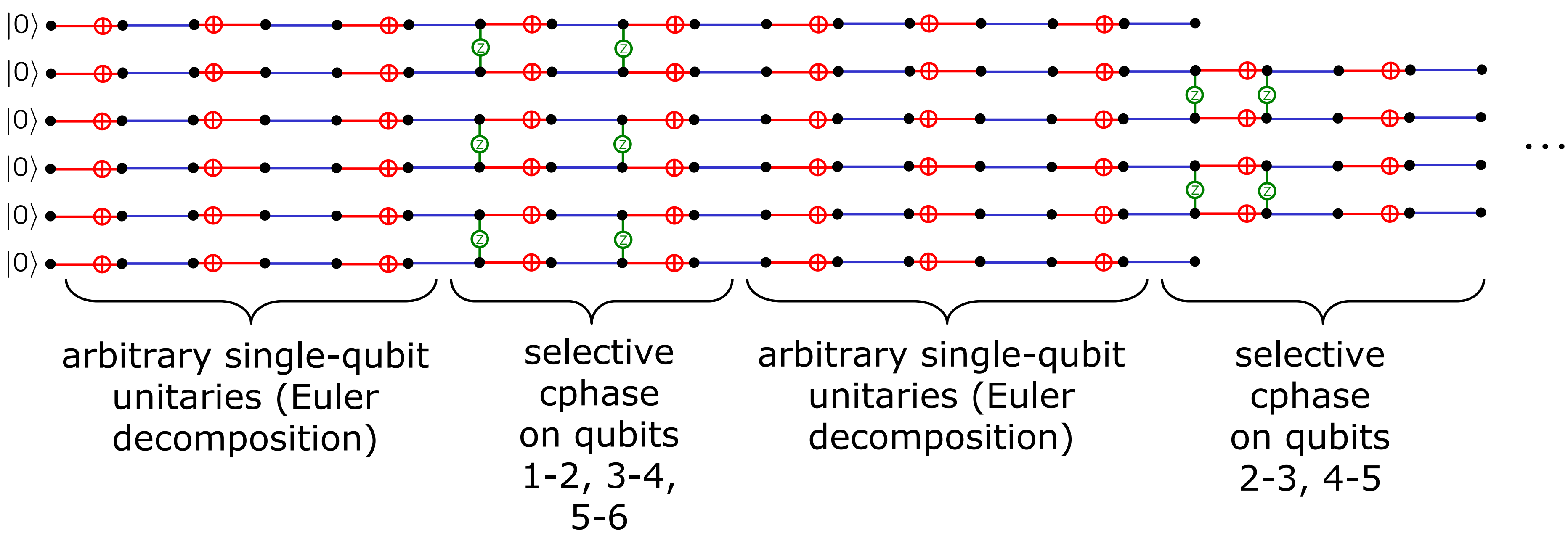
Teleportation circuit for cancellation approach



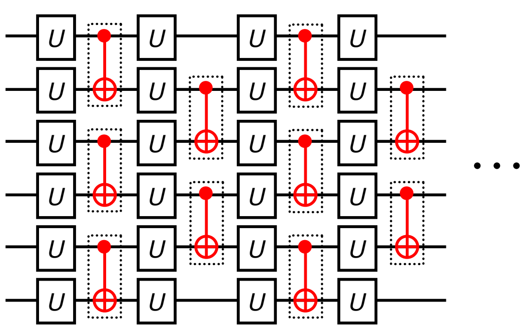
Teleportation circuit for cancellation approach



Universal state for cancellation approach



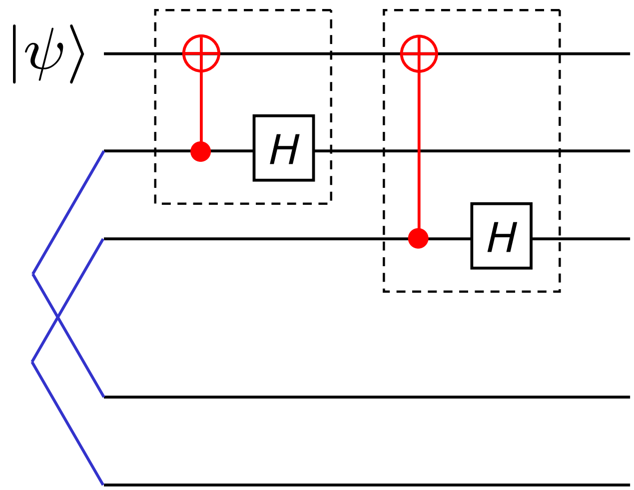
Using single-qubit measurements, this state can be used to implement a circuit of the form

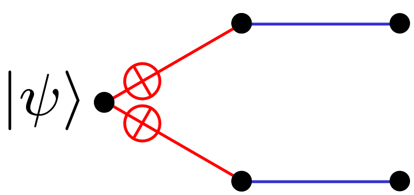


which is clearly universal.

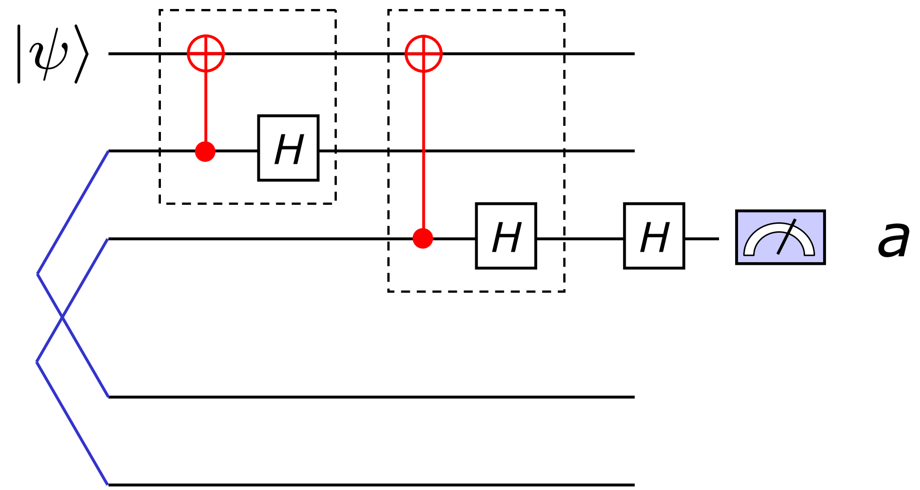
Routing approach

Move a logical qubit to different physical qubits by choice of measurement.

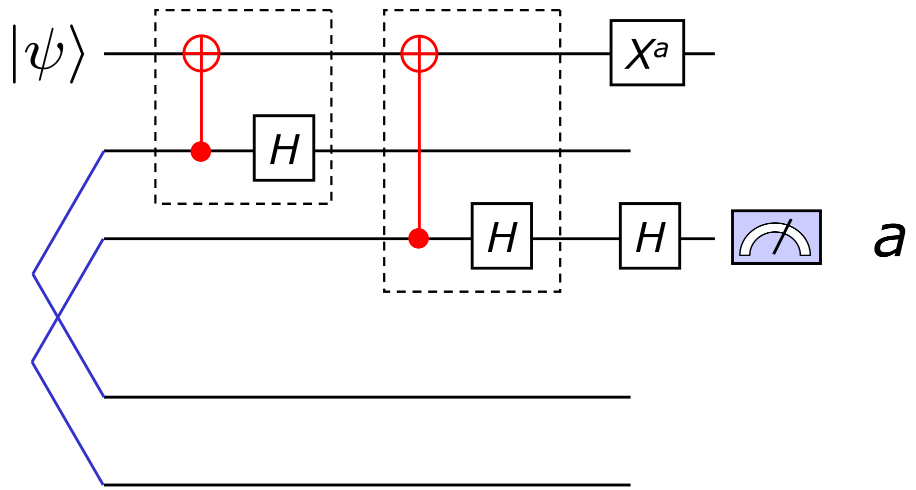




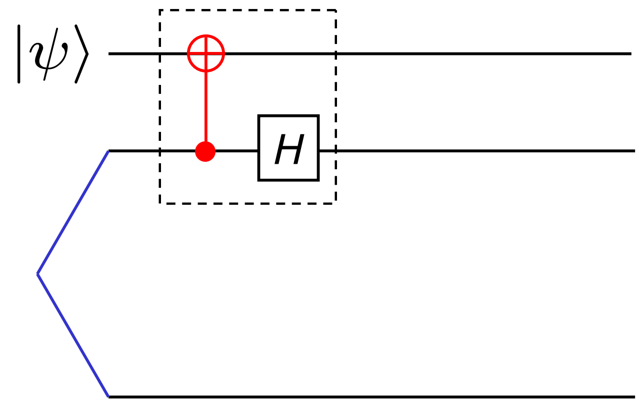
Routing approach



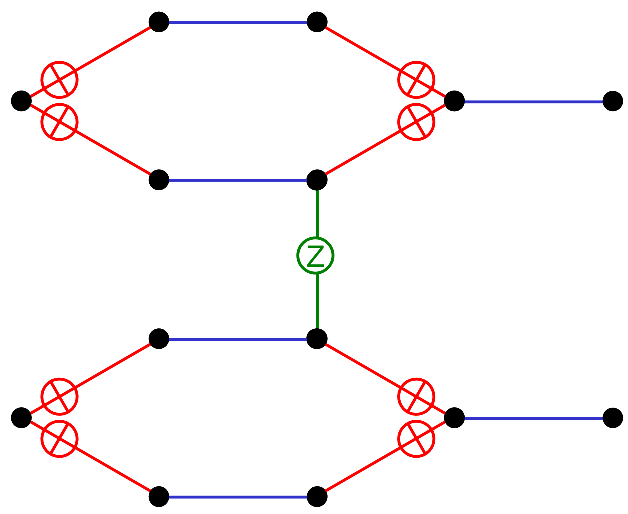
Routing approach



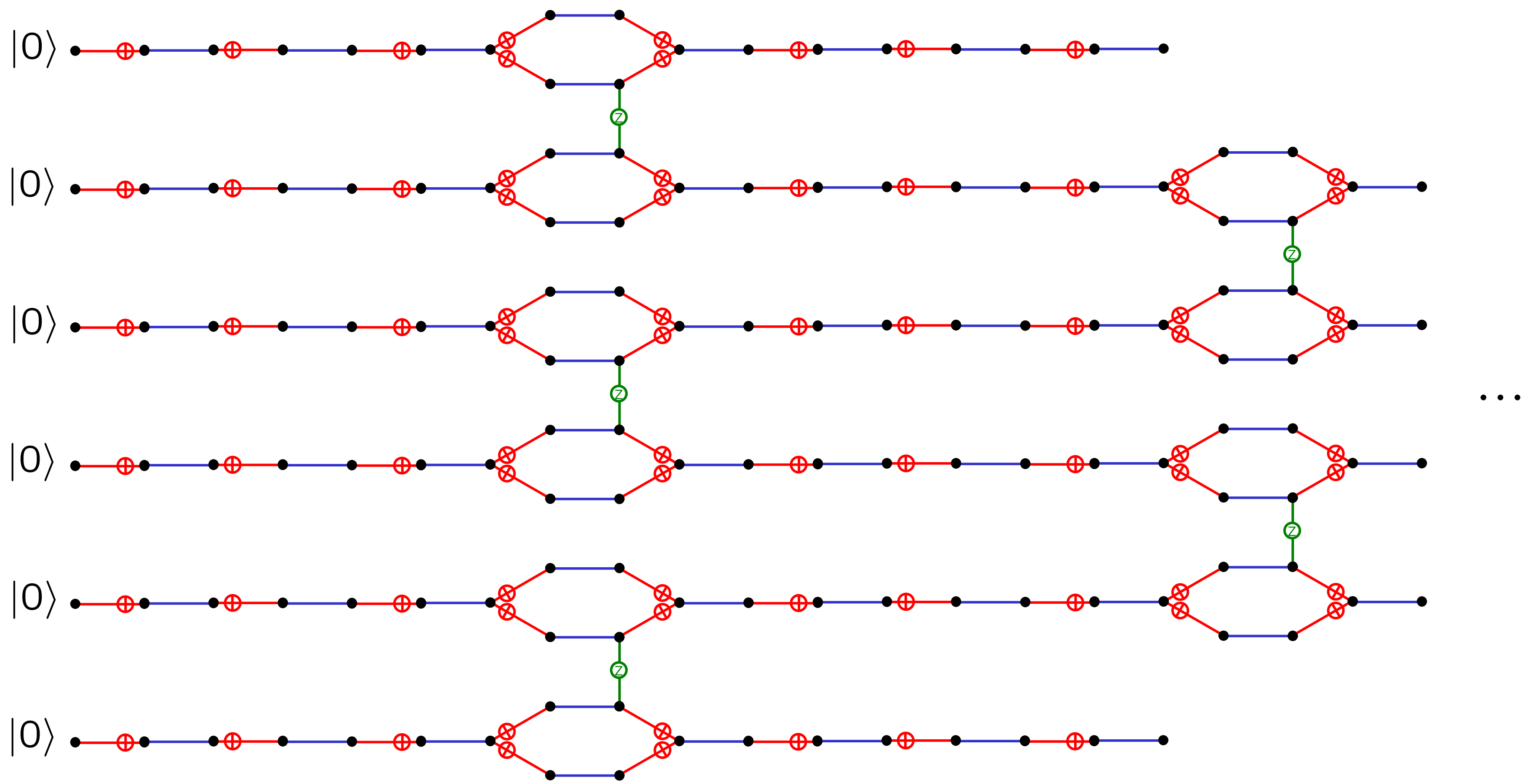
Routing approach



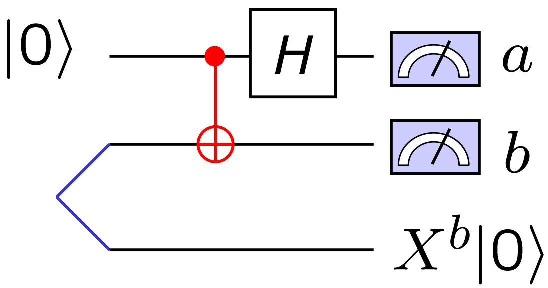
Selective interaction by routing



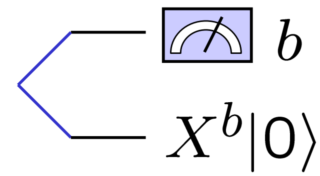
Universal state for routing approach



State preparation and readout



is extraneous; instead:



Final state is $X^{a_1}Z^{b_1} \otimes \cdots \otimes X^{a_n}Z^{b_n}|\psi\rangle$, where $|\psi\rangle$ is the desired state, and $a_1, b_1, \ldots, a_n, b_n$ are known.

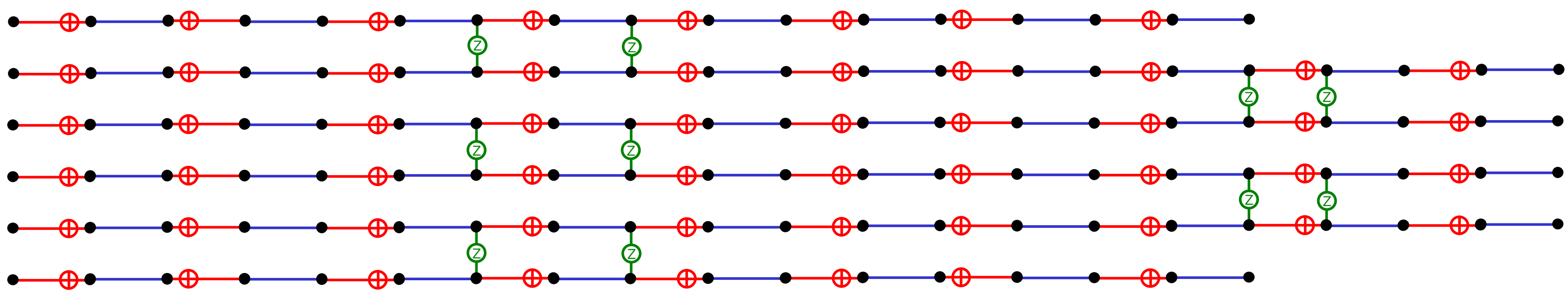
Measure in the computational basis and flip the bits j for which $a_j=1$.

States for universal quantum computation by measurement

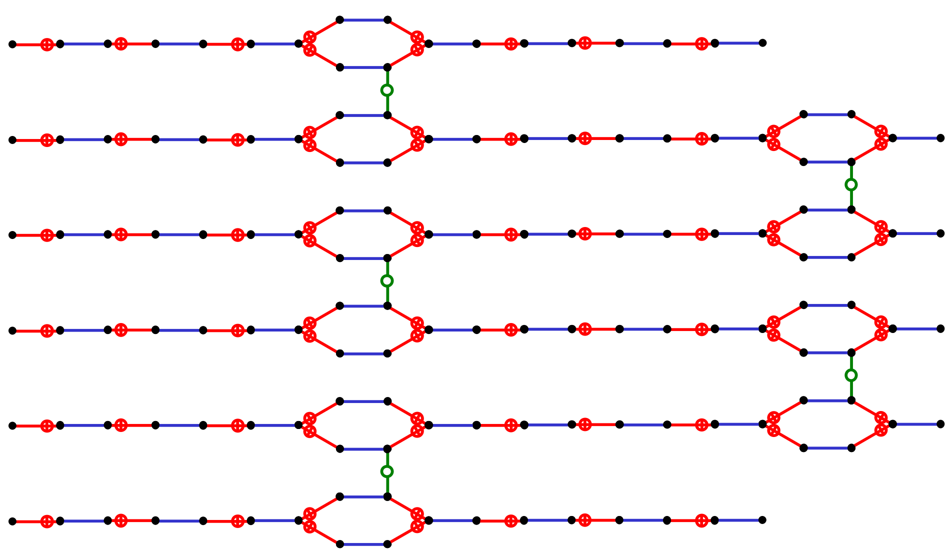
Two families of states $|C_{n,m}\rangle$ and $|R_{n,m}\rangle$ where

- n = number of logical qubits
- m = number of computational steps

$|C_{6,1}\rangle =$



$|R_{6,1}\rangle =$



Stabilizer

Stabilizer of $|\psi\rangle$: $S_{|\psi\rangle} = \{U \in C_1 : U|\psi\rangle = |\psi\rangle\}$

S  $= \langle XX, ZZ \rangle$

$$IX \rightarrow IX$$
$$XI \rightarrow XX$$
$$IZ \rightarrow ZZ$$
$$ZI \rightarrow ZI$$

$$IX \rightarrow ZX$$
$$XI \rightarrow XZ$$
$$IZ \rightarrow IZ$$
$$ZI \rightarrow ZI$$

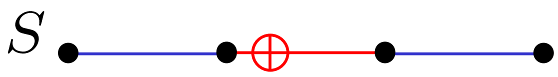
$$S_{|\psi\rangle \otimes |\phi\rangle} = \langle S_{|\psi\rangle} \otimes I, I \otimes S_{|\phi\rangle} \rangle$$

Stabilizer

S  $= \langle XXII, IIXX, ZZII, IIZZ \rangle$

weight=2

Stabilizer



$$= \langle XXII, IXXX, ZZZI, IIZZ \rangle$$

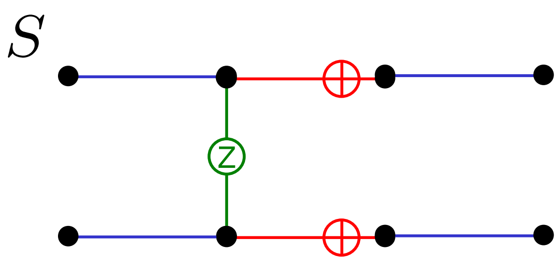
weight=3

$$= \langle XXIIII, IXXXII, IIIXXX, ZZZIII, IIZZZI, IIIIZZ \rangle$$

weight=3

$$= \langle XXIIII, IXXXXI, IIIIXX, ZZZIII, IIZZII, IIIZZZ \rangle$$

weight=4



$$= \left\langle \begin{matrix} X\,X\,X\,I \\ I\,I\,Z\,I \end{matrix}, \begin{matrix} I\,I\,X\,X \\ I\,I\,Z\,I \end{matrix}, \begin{matrix} Z\,Z\,I\,I \\ I\,I\,I\,I \end{matrix}, \begin{matrix} I\,Z\,Z\,Z \\ I\,I\,I\,I \end{matrix}, \begin{matrix} I\,I\,Z\,I \\ X\,X\,X\,I \end{matrix}, \begin{matrix} I\,I\,Z\,I \\ I\,I\,X\,X \end{matrix}, \begin{matrix} I\,I\,I\,I \\ Z\,Z\,I\,I \end{matrix}, \begin{matrix} I\,I\,I\,I \\ I\,Z\,Z\,Z \end{matrix} \right\rangle$$

weight=4

$\Rightarrow$ Weight of $|C_{n,m}\rangle$ is 4. Can prepare $|C_{n,m}\rangle$ from an unentangled state using 4-qubit measurements.

Fault tolerance?

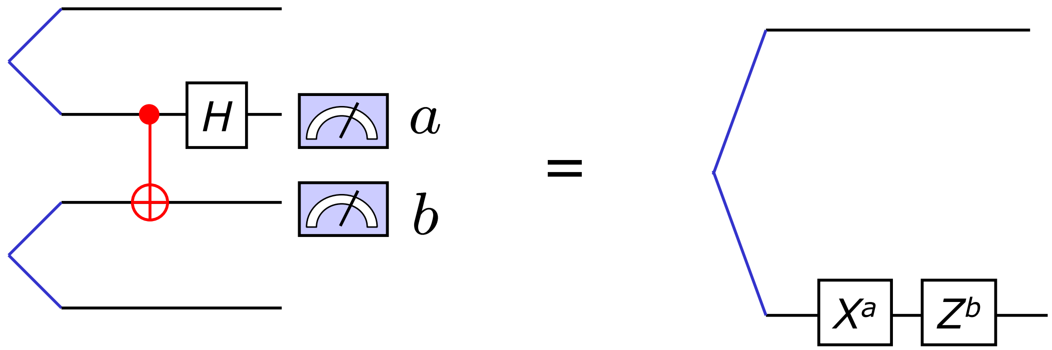
Simple noise model: At each time step, each qubit has a probability p of undergoing a random Pauli error.

In the circuit model, there is a fault tolerant threshold (Shor; Aharonov and Ben-Or). If (say) $p<10^{-4}$, we can implement an arbitrarily long computation with arbitrarily small probability of error.

Fundamental problem: We must be able to prepare fresh qubits (cf. Aharonov, Ben-Or, Impagliazzo, Nissan) and subsequently use them in our computation. But this cannot be done using only single-qubit measurements!

One way out: Allow occasional two-qubit measurements. (Think of them as more like state preparation steps than computational steps.)

Entanglement swapping



$\Rightarrow$ We can join our universal states using Bell measurements.

$$|C_{n,m}\rangle + |C_{n,m'}\rangle + n \text{ Bell measurements} \rightarrow |C_{n,m+m'}\rangle$$

(and similarly for $|R_{n,m}\rangle$)

Open questions

- How do these schemes relate to the "cluster state" quantum computer of Raussendorf and Briegel?
- Are such approaches useful for building a quantum computer?
- Can these schemes be made fault tolerant in some reasonable model?
- If so, can we find a threshold that is competitive with, or perhaps even better than, the known circuit model thresholds?