Introduction to Quantum Mechanics

QCSYS 2011

Outline

Polarization

Superposition

A basic feature of quantum mechanics is the *principle of superposition*:

If a quantum system can be in the state $|\psi\rangle$ or in the state $|\phi\rangle$, then it can also be in state $\alpha|\psi\rangle + \beta|\phi\rangle$ for any complex numbers α, β (subject to normalization).

Example:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The superposition principle is also shared by classical waves.

We'll explore superposition in the context of *polarization* of light.

The electromagnetic field

In classical electromagnetism, there is an

electric field $\vec{E}(x, y, z, t)$

magnetic field $\vec{B}(x, y, z, t)$

at every spacetime point (x, y, z, t).

These fields obey the *Maxwell equations*:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad\qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad\qquad \vec{\nabla} \times \vec{B} = \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$
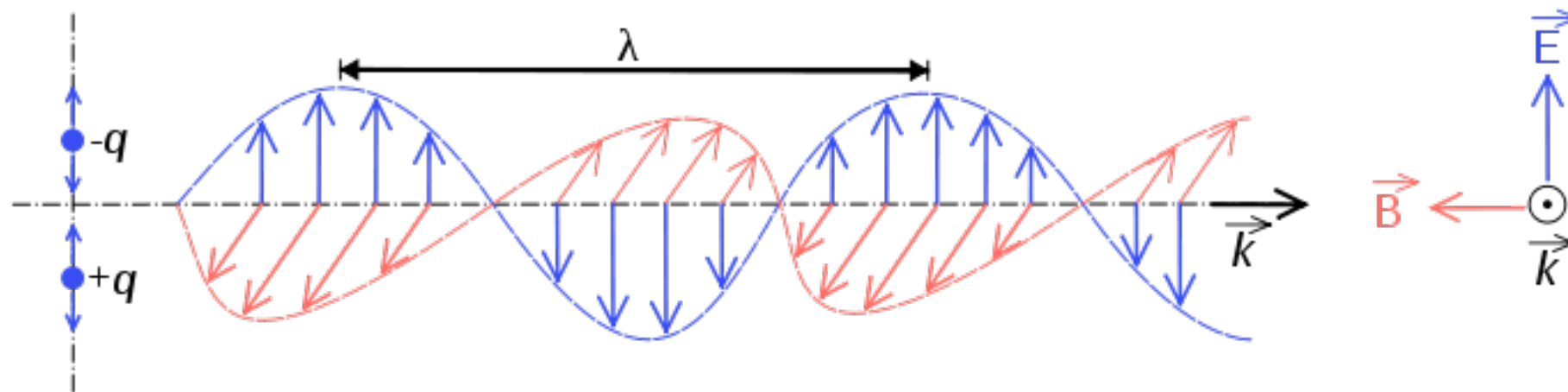
Traveling waves

The Maxwell equations have solutions that correspond to waves propagating through space.

Example: Plane wave propagating in the $\hat{z}$ direction

$$\vec{E}(x, y, z, t) = \mathrm{Re}\left(\begin{pmatrix} \alpha_x \\ \alpha_y \\ 0 \end{pmatrix} e^{2\pi i(z - ct)/\lambda}\right)$$

$$\vec{B}(x, y, z, t) = \hat{z} \times \vec{E}(x, y, z, t)$$

The vector $\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}$ indicates the *polarization* of the wave.

Superposition of polarization states

Horizontal polarization: $|\rightarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Vertical polarization: $|\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

45° diagonal polarization (normalized states):

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\searrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle - \frac{1}{\sqrt{2}}|\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The 45° states also form an orthonormal basis:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle + \frac{1}{\sqrt{2}}|\searrow\rangle$$

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle - \frac{1}{\sqrt{2}}|\searrow\rangle$$

Polarizing filters

Most sources of light are unpolarized.

We can create polarized light using a filter that only allows one of two orthogonal polarizations to pass.

Mathematically, this implements a *projection* onto a the polarization direction of the filter.

- The component along the filter direction passes through.
- The component orthogonal to the filter direction is blocked.

Analogous to quantum measurement:

- The fraction of light that passes through the filter is given by the inner product squared with the filter direction.
- The outgoing light is entirely in the same direction as the filter.

Polarizing filter examples

Incident polarization: $\left|\nearrow\right\rangle$

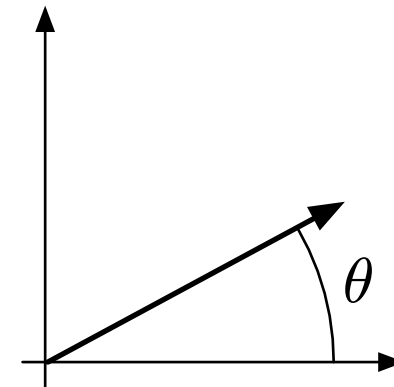Polarizing filter orientation: $\rightarrow$

How much light passes? 50%

Outgoing polarization? $\left|\rightarrow\right\rangle$

Incident polarization: $\left|\rightarrow\right\rangle$

Polarizing filter orientation: $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

How much light passes? $\cos^2\theta$

Outgoing polarization? $\cos\theta\left|\rightarrow\right\rangle + \sin\theta\left|\uparrow\right\rangle$

Polarizing filter demo

What happens to the incident light if we orient polarizers as follows?

- crossed polarizers (0 and 90 degrees)
- diagonal polarizers (0 and 45 degrees)
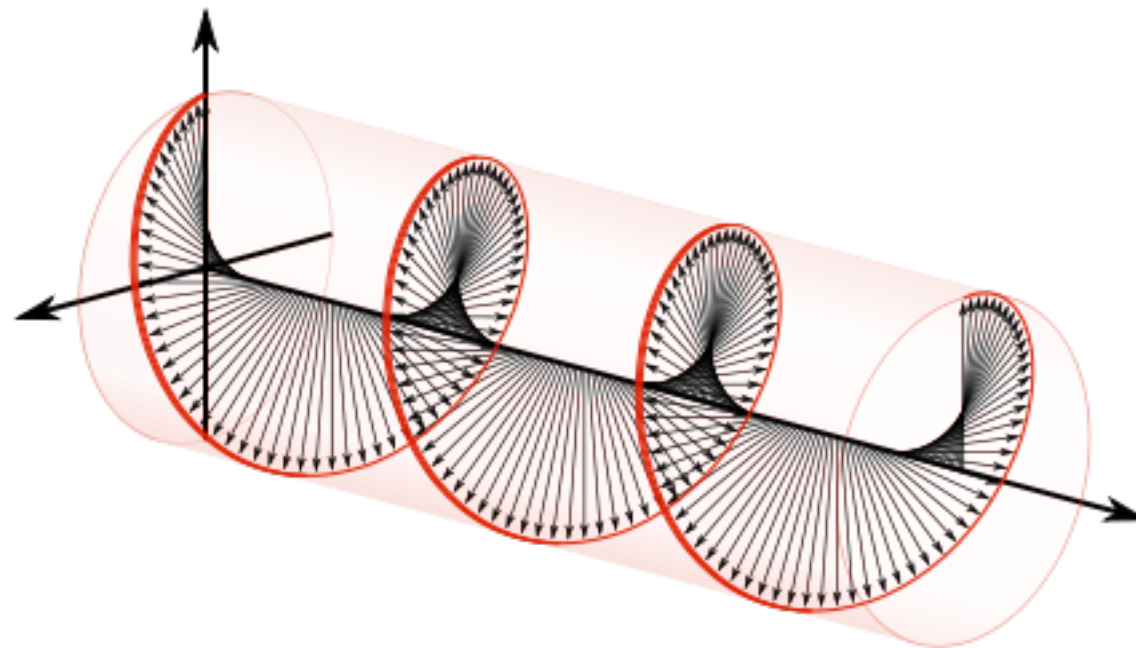- polarizers at 0, 45, 90 degrees

Circular polarization

We can also consider superpositions involving complex numbers.

Examples:

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle + i|\uparrow\rangle) \qquad |\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle - i|\uparrow\rangle)$$

The direction of the electric field moves in a circle as the wave propagates, so this is called *circular polarization*.

Exercise: Stacked polarizers

Suppose we stack n polarizers so that the angle between the polarization direction of each filter and the next is π/n. What fraction of the light passing the first polarizer passes the last polarizer?
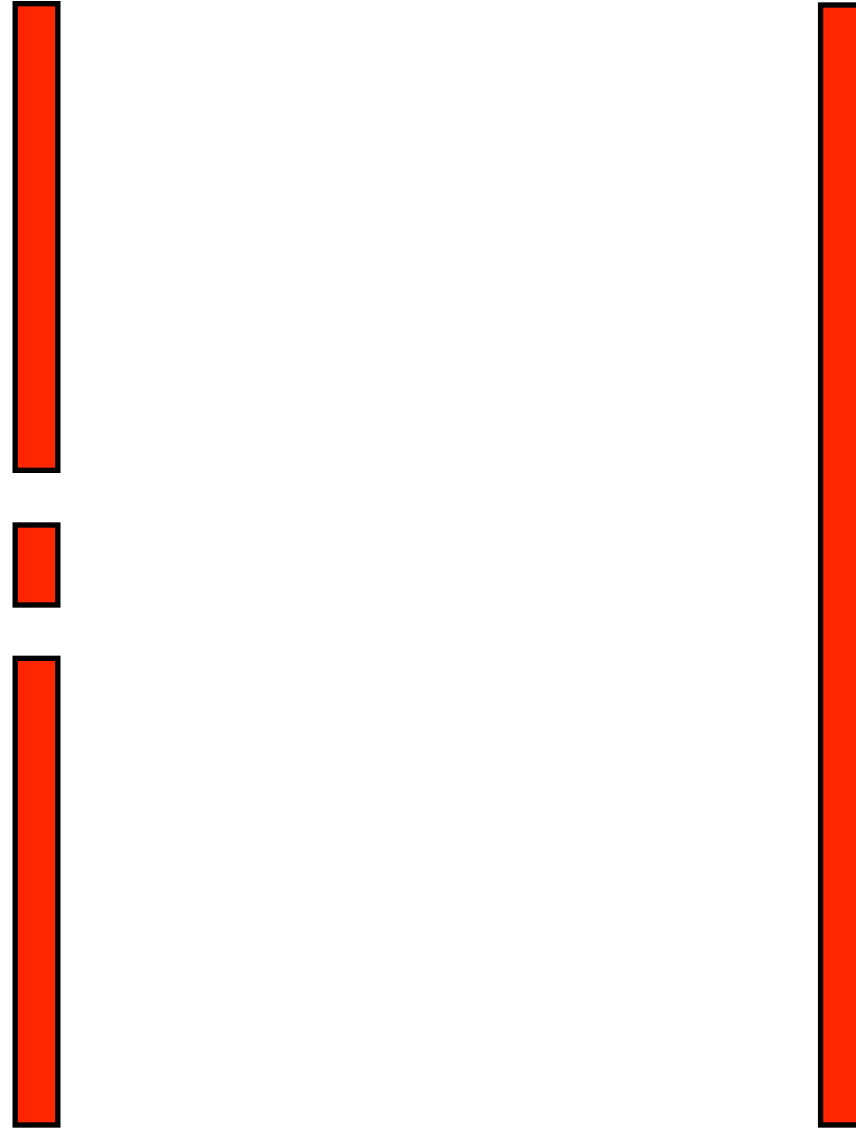
   a. Compute exact values for $n = 2, 3, 4$.

   b. Give a symbolic expression for general n.

   c. Using a computer, plot the values for $n = 2$ through 50.

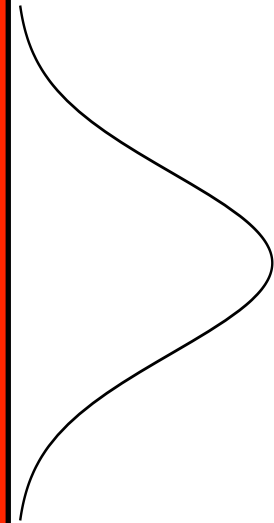   d. What happens in the limit as $n \to \infty$?

Double-slit experiment

Firing bullets at a slit
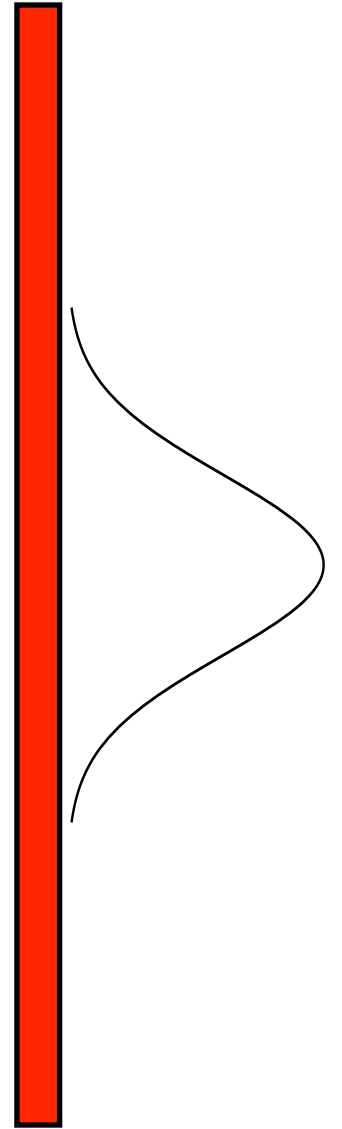
Firing bullets at a double slit
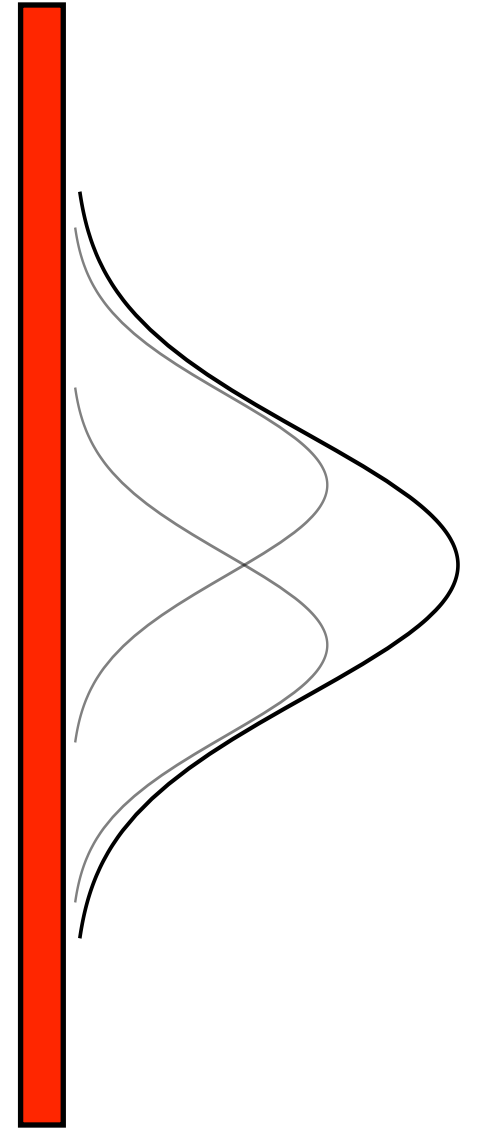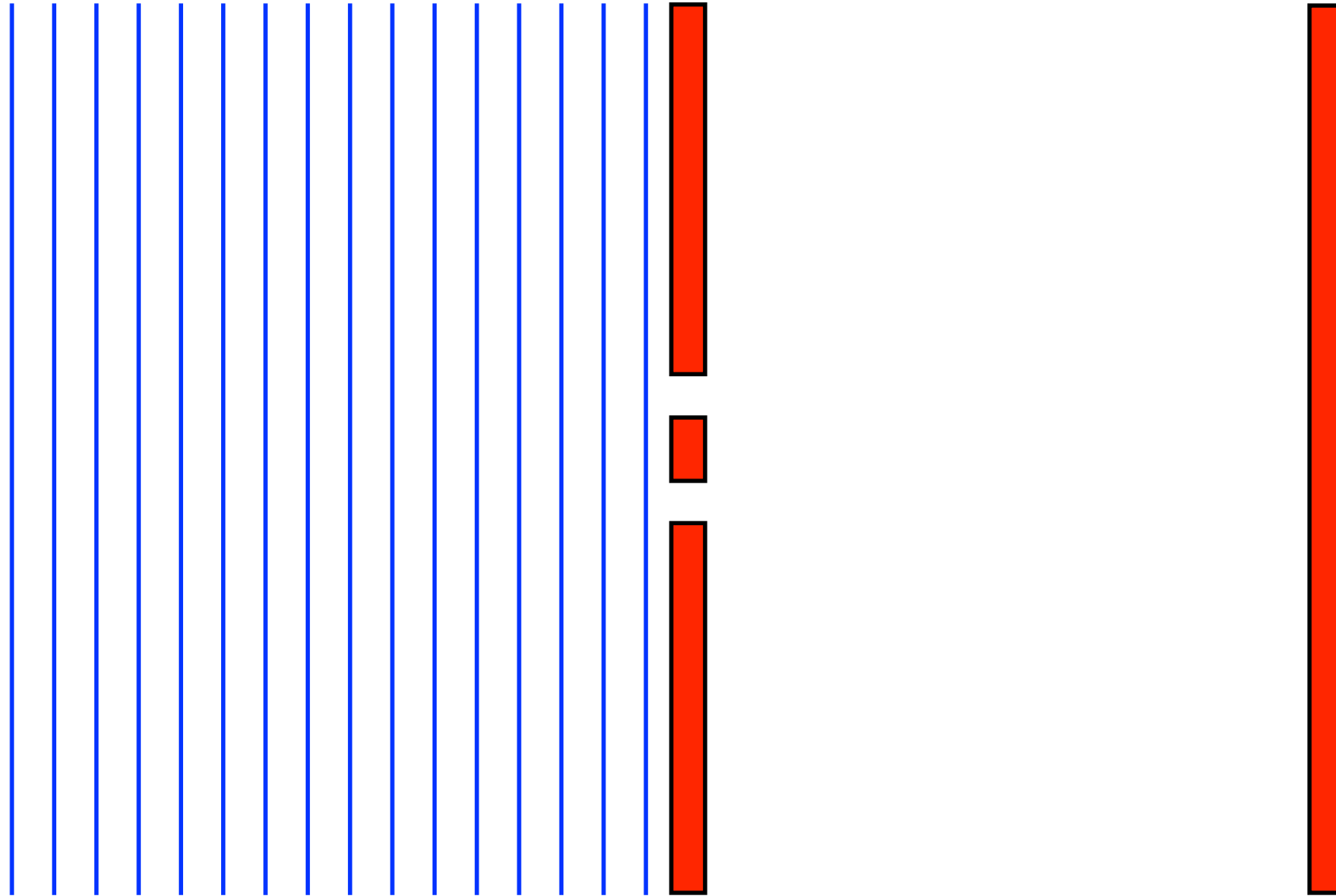
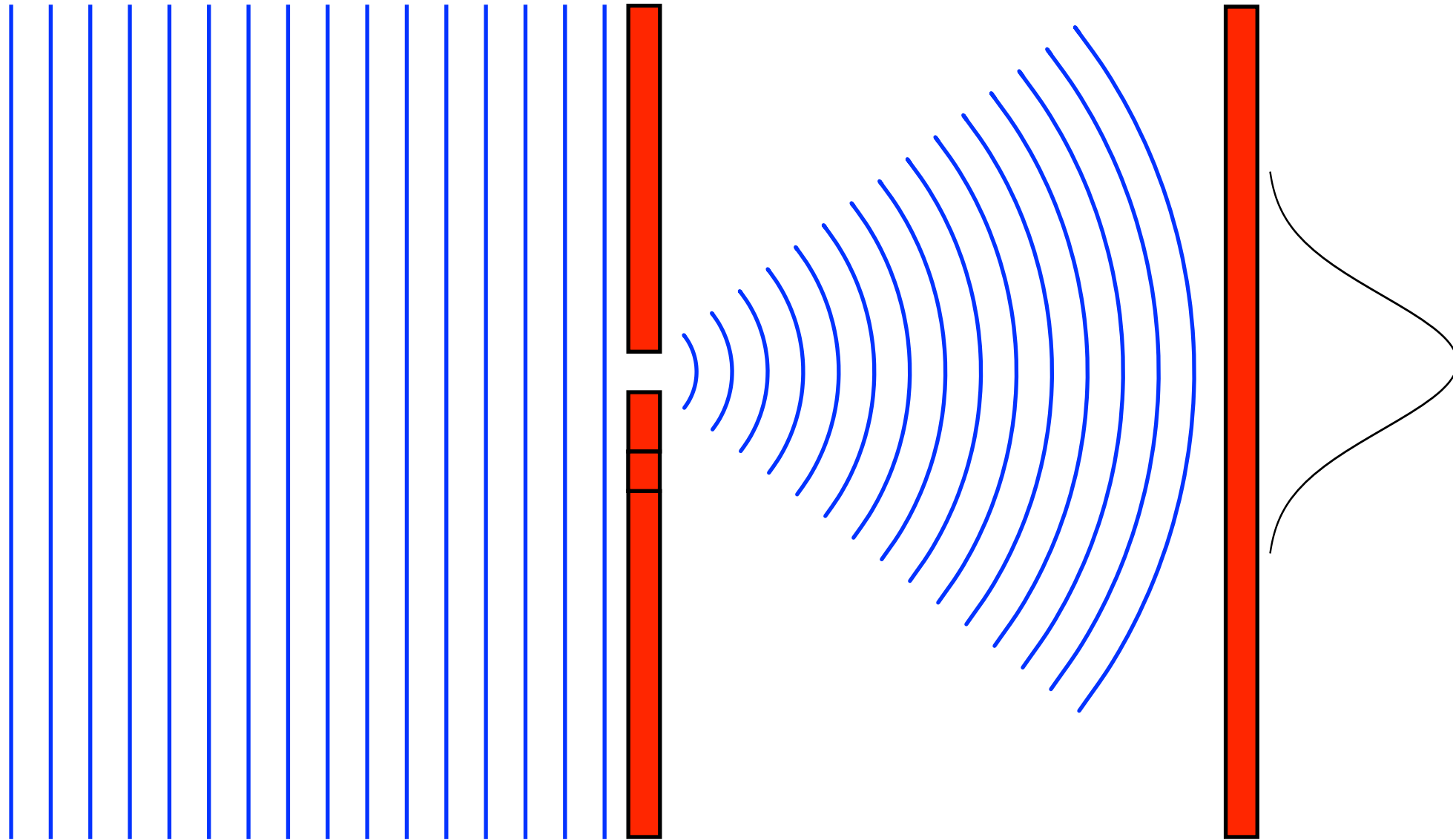# Firing bullets at a double slit

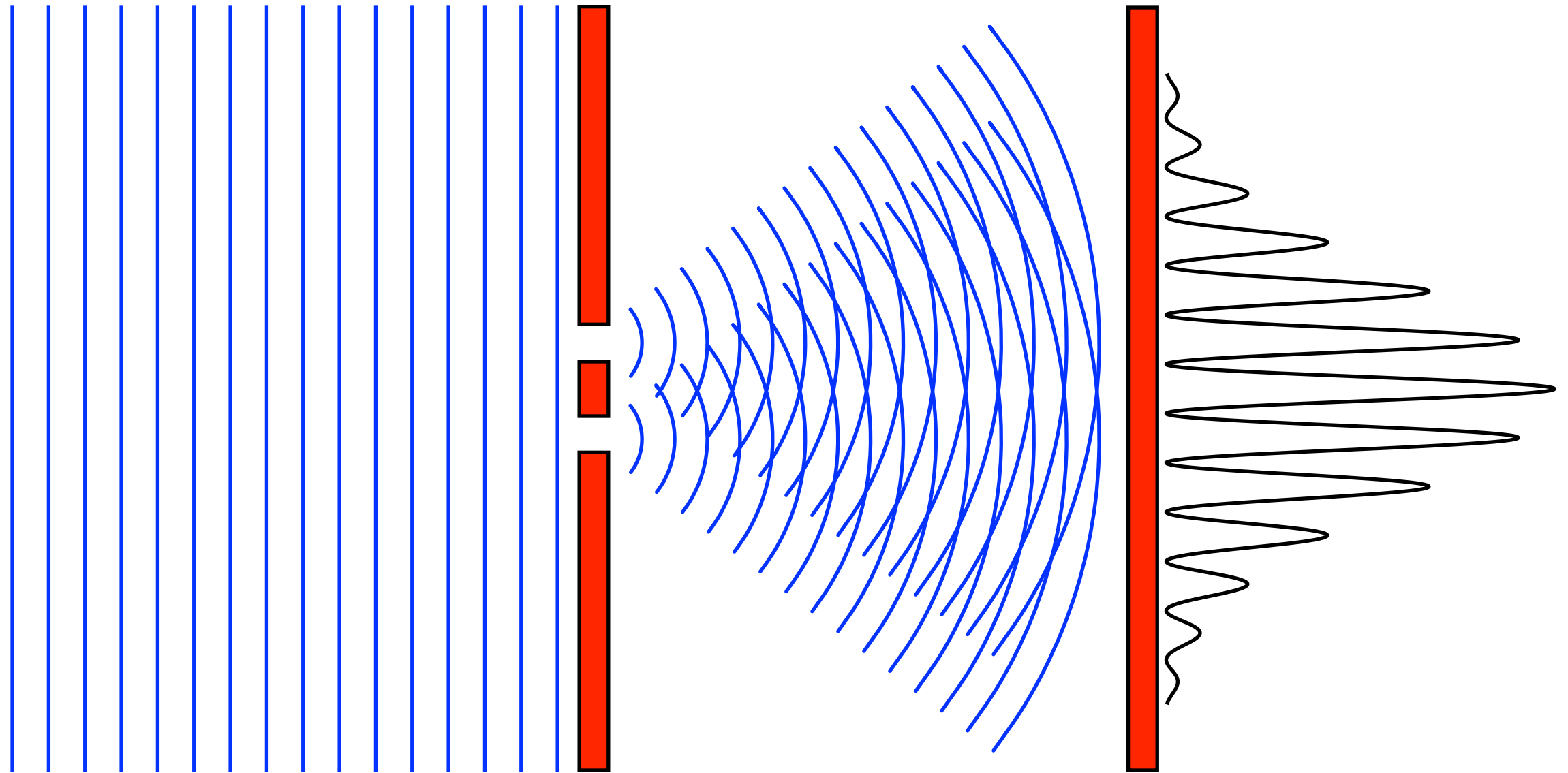# Firing bullets at a double slit

Firing bullets at a double slit

Double slit with waves

Double slit with waves

Double slit with waves

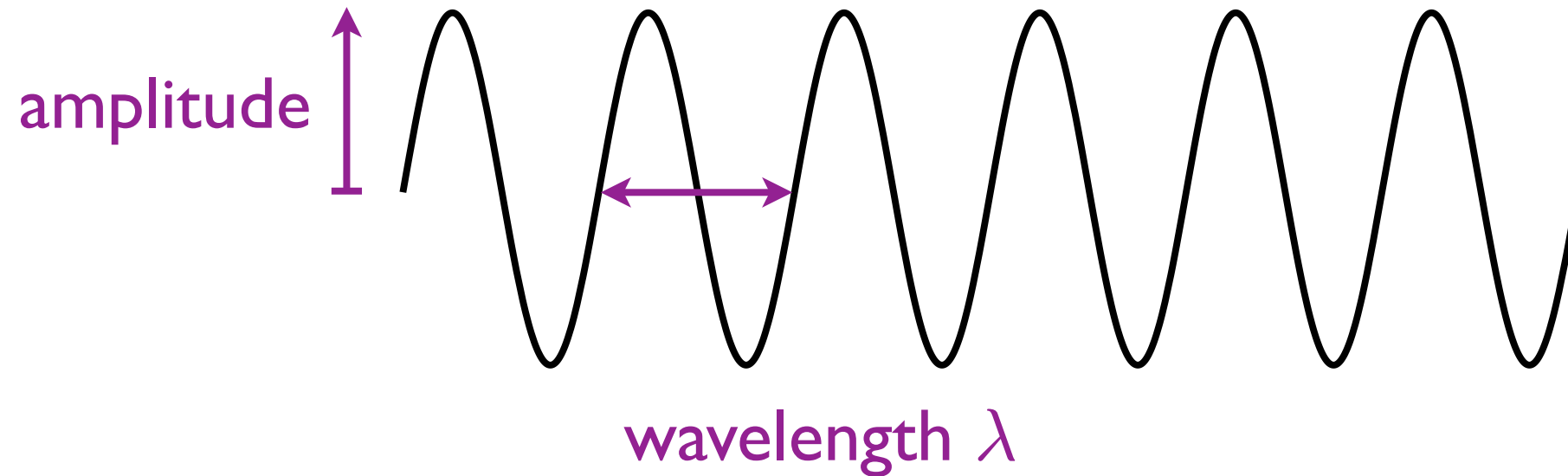# Demo

Interference

amplitude

wavelength λ
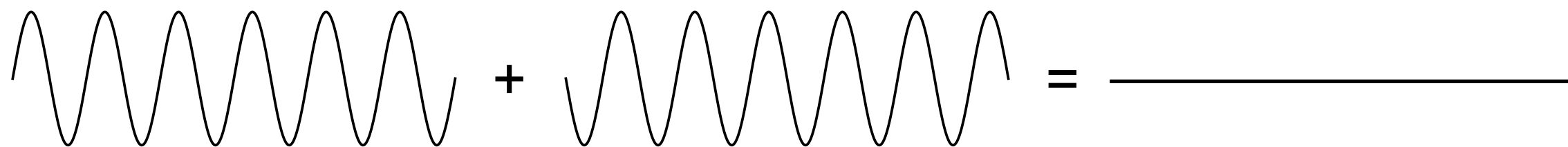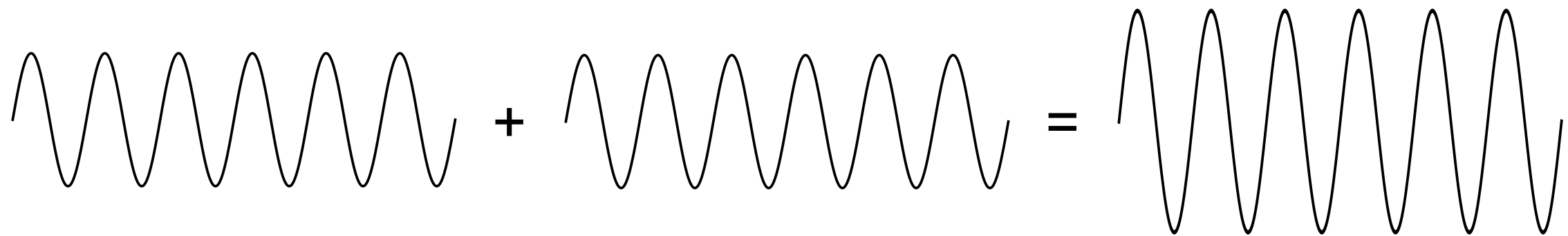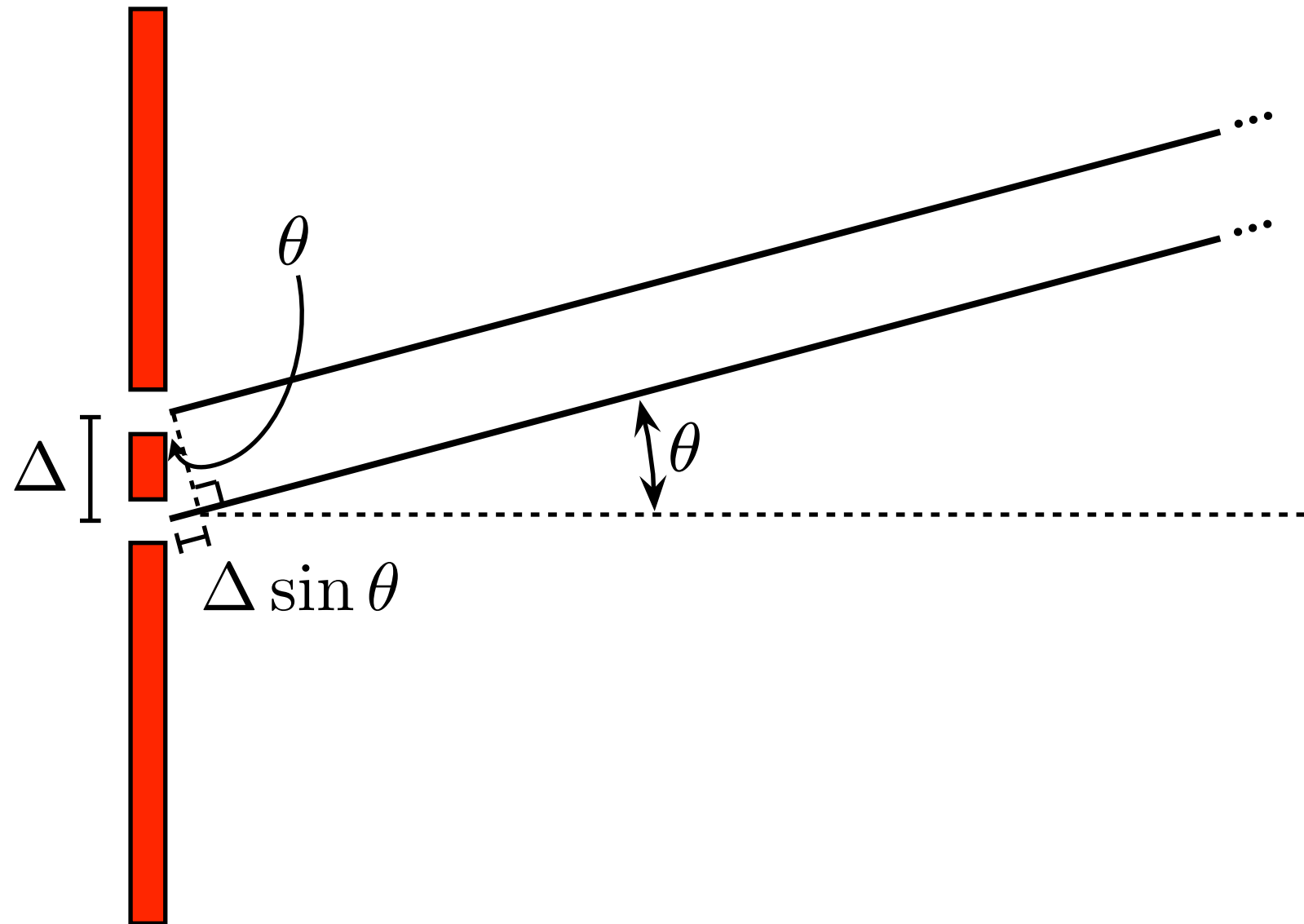
Amplitude can be positive (water is above sea level)
or negative (water is below sea level)

Form of the interference pattern

One fringe: $\Delta \sin \theta = \lambda$ $\theta \approx \dfrac{\lambda}{\Delta}$

Screen at a distance d away: fringe spacing is approximately $d \cdot \theta \approx \dfrac{d \cdot \lambda}{\Delta}$

Double slit with electrons

How will the experiment behave if we use electrons instead of bullets or water waves?

- Electrons come in discrete chunks, like bullets.
- Nevertheless, the experiment shows an interference pattern!

What does this mean? Is an electron a particle or a wave?

- Yes. ("wave-particle duality")

What if we observe which slit the electrons go through?

Same behavior with light, which is composed of individual photons.

Uncertainty principle

In classical mechanics, nothing prevents us from measuring the state of a particle (its position and momentum) with arbitrary precision.

Quantum mechanics forbids this: it places fundamental limitations on the kinds of measurements that can be carried out.

Heisenberg uncertainty principle: $\Delta x \, \Delta p \geq \dfrac{\hbar}{2}$

# Uncertainty principle and diffraction
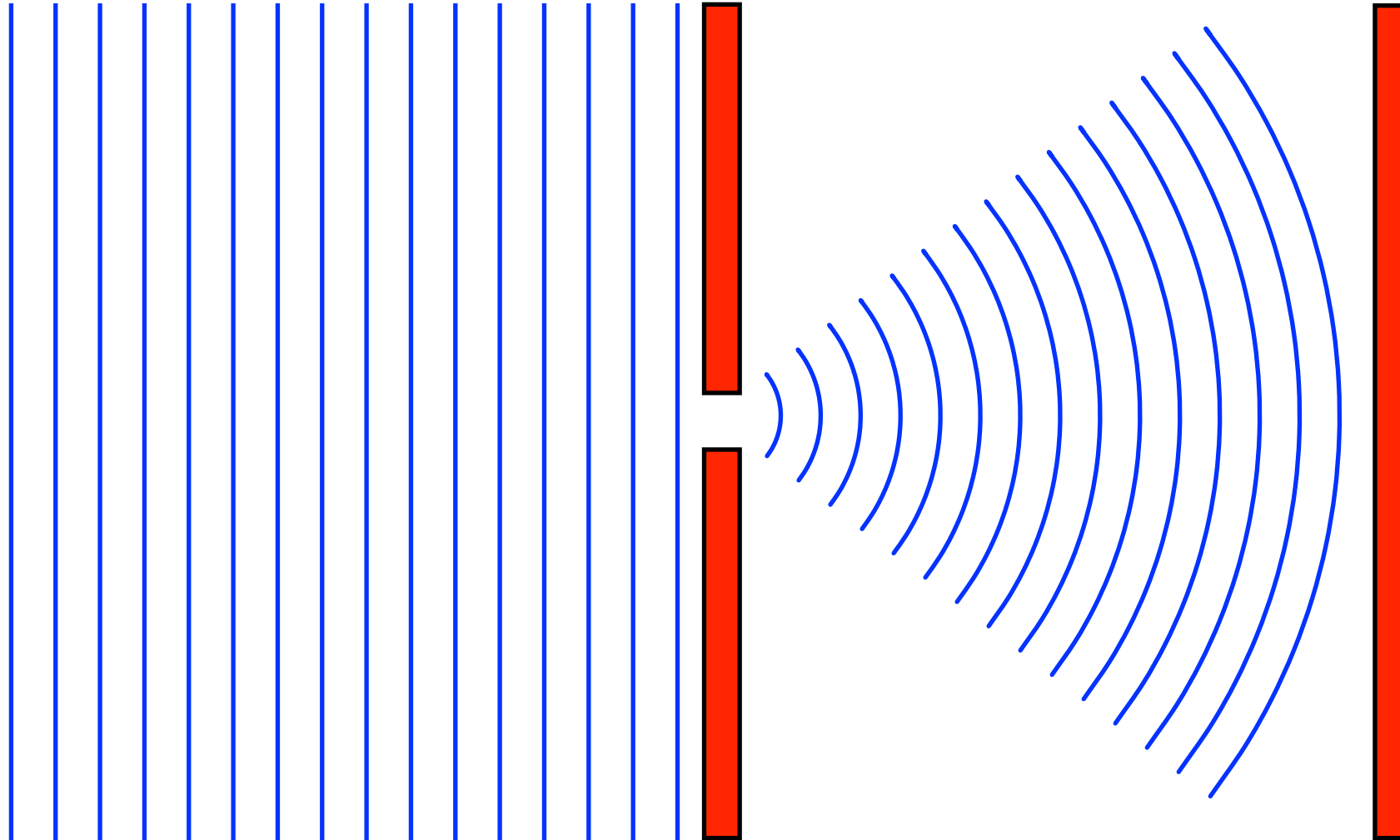
Exercise: Double slit with laser light

Suppose you perform the double slit experiment using a green laser with a wavelength of 523 nm and slits spaced by 1 mm.

a. What is the angular spacing between two adjacent fringes of the interference pattern?

b. If the pattern is projected onto a screen at a distance of 5 m, what is the distance between adjacent fringes?

Photoelectric effect

The photoelectric effect
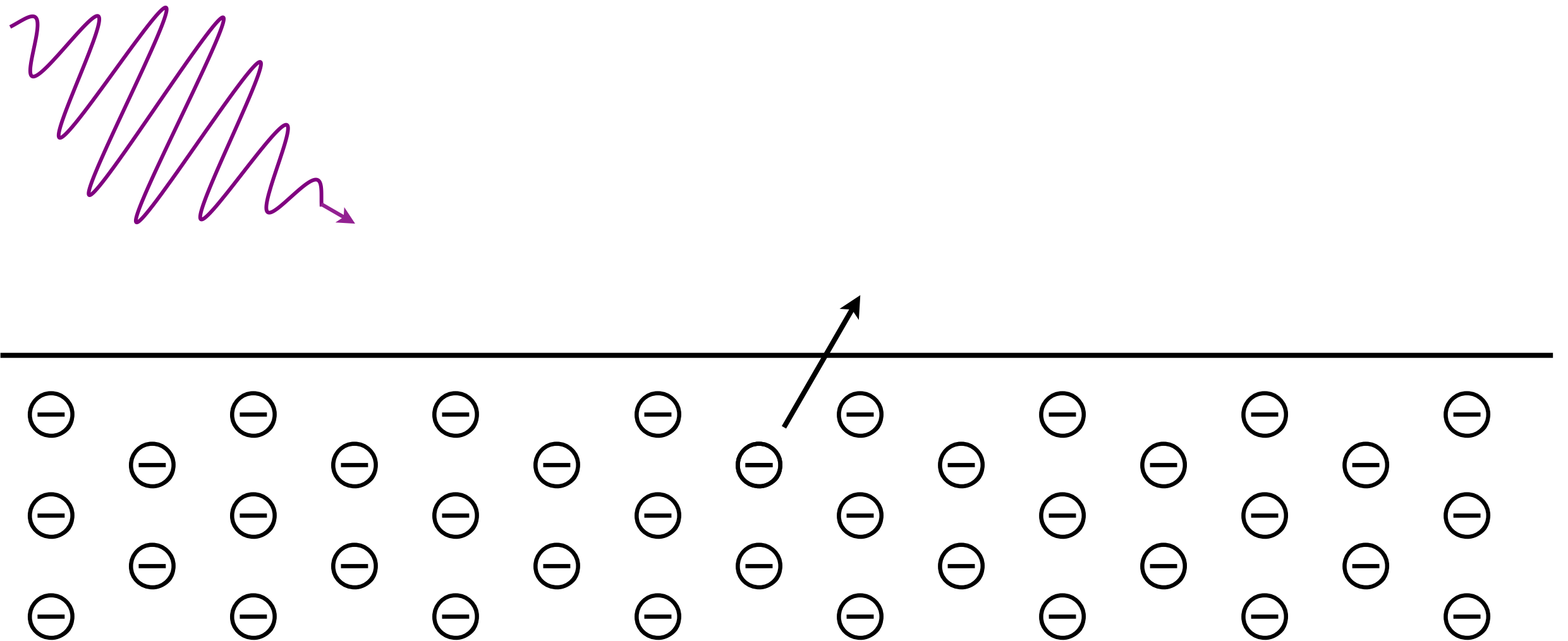
Photons and electrons

Light is made up of massless particles called photons.

Photons are characterized by their wavelength λ
or equivalently, their frequency ν.

$$E = h\nu = \frac{hc}{\lambda}$$

$$h = 6.62 \times 10^{-34}\,\mathrm{J \cdot s}$$

$$c = 3 \times 10^{8}\,\mathrm{m/s}$$

Electrons are massive particles with negative electric charge.

$$E = \frac{1}{2}mv^2 + \phi$$

$$m = 9.11 \times 10^{-31}\,\mathrm{kg}$$

$$e = 1.60 \times 10^{-19}\,\mathrm{C}$$

The work function

Removing an electron from a material costs energy.

In the context of the photoelectric effect, this is called the *work function*, denoted ϕ. Its value depends on the material. Typical values are a few electron volts.
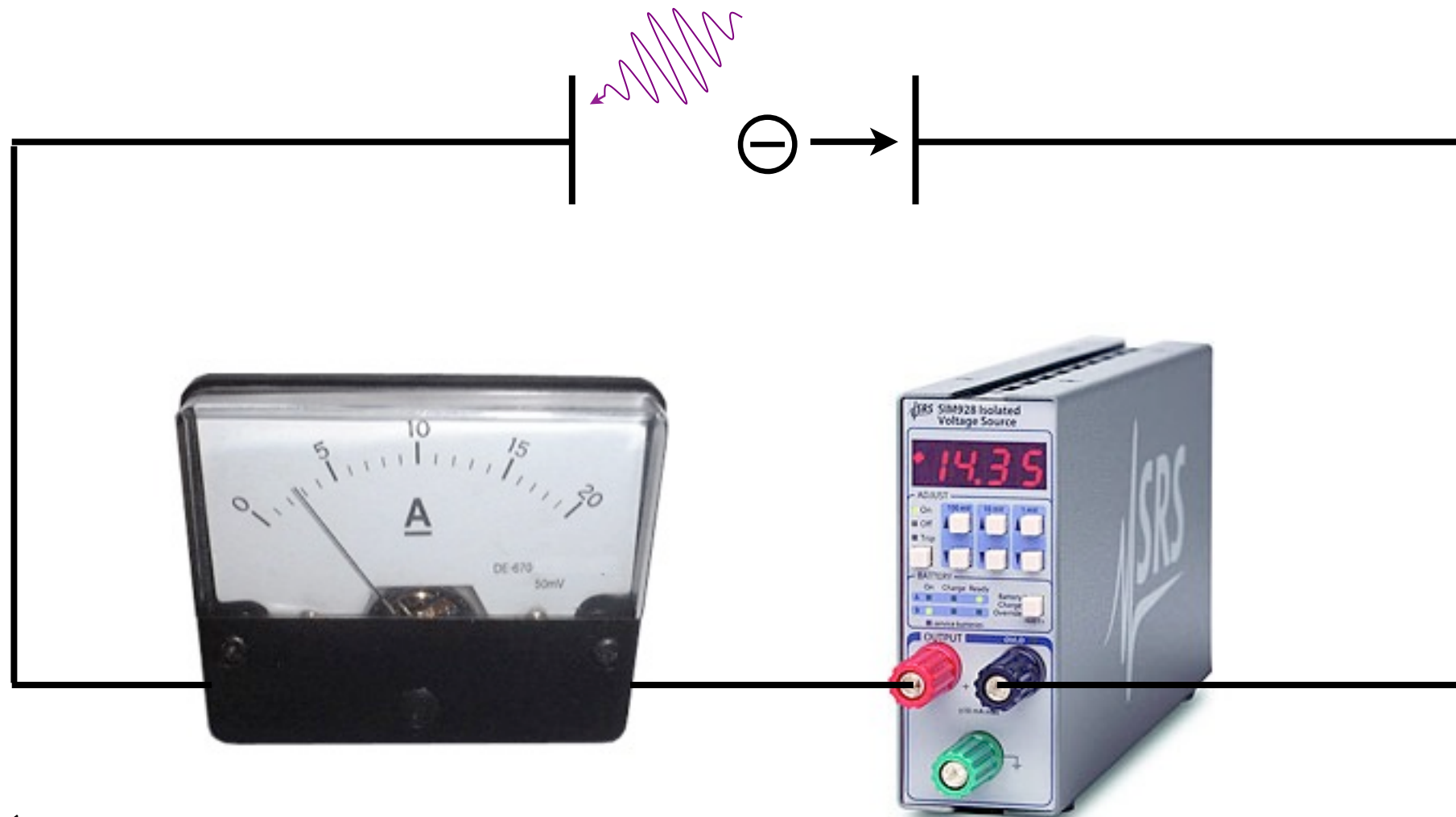
By conservation of energy, for a photon of frequency ν to eject an electron with velocity v, we have

$$h\nu = \frac{1}{2}mv^2 + \phi$$

Whether emission occurs depends on the frequency of the light, not on its intensity (the number of photons arriving per unit time).
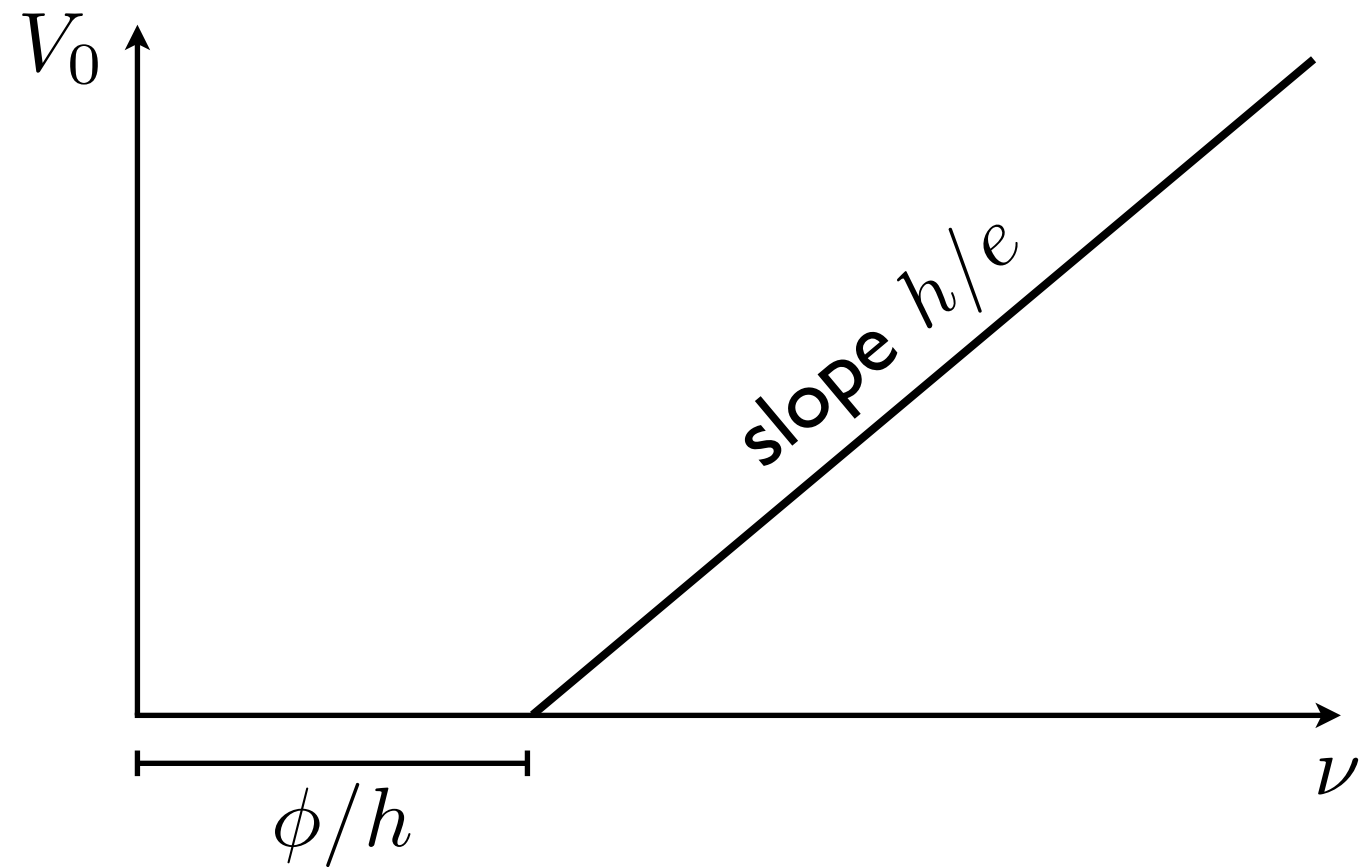
Photoelectric effect experiment

$$h\nu = \frac{1}{2}mv^2 + \phi$$

$$\frac{1}{2}mv^2 = eV_0$$ where V_0 is the *stopping potential*, the voltage that must be applied to stop the current from flowing

Photoelectric effect experiment: results

$$h\nu = \frac{1}{2}mv^2 + \phi$$

$$\frac{1}{2}mv^2 = eV_0$$

$$V_0 = \frac{h\nu - \phi}{e}$$

Exercise: Photoelectric effect in platinum

For this problem, the following values may be useful:

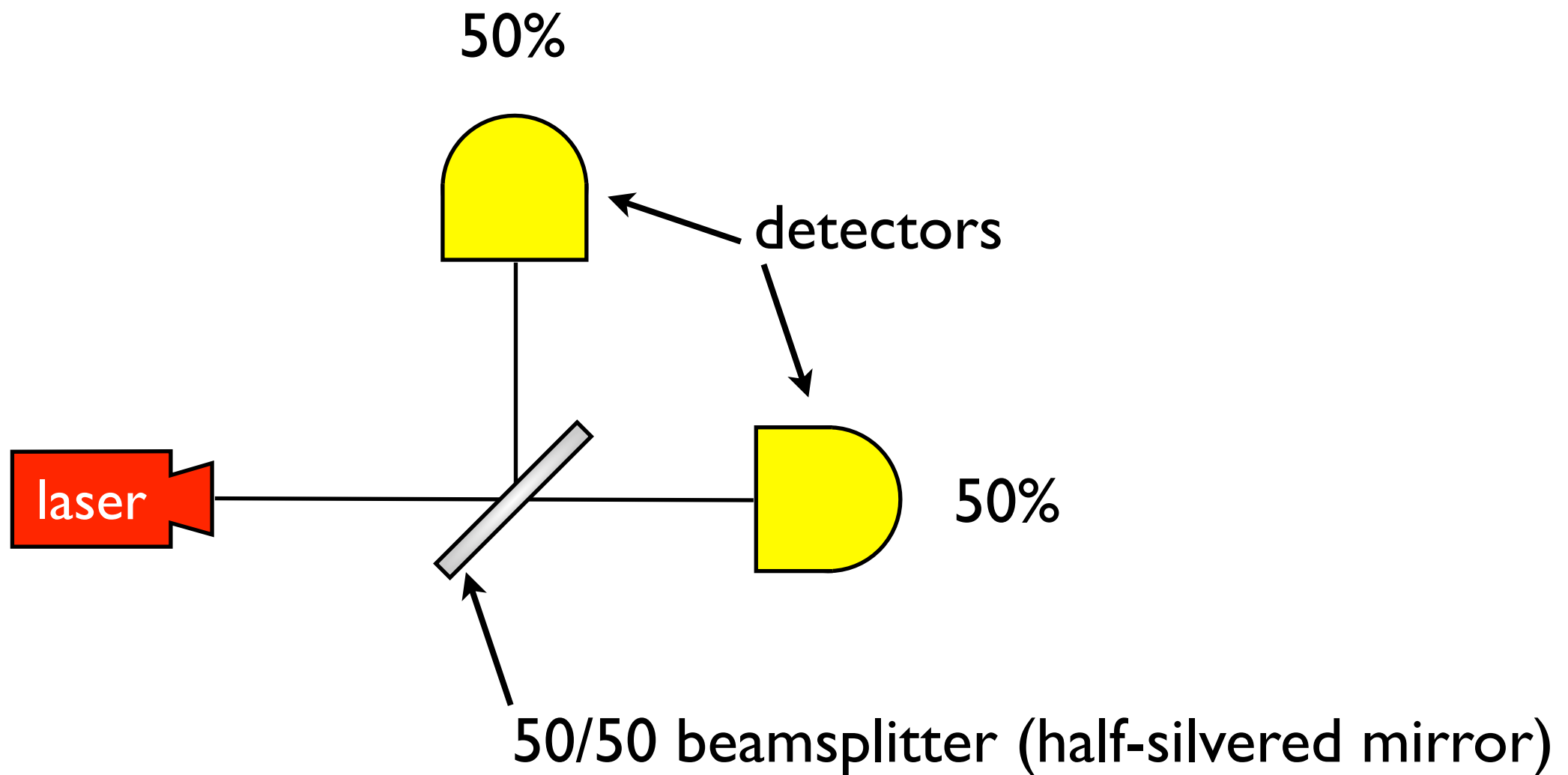$$h = 4.14 \times 10^{-15} \ \text{eV} \cdot \text{s}$$

$$c = 3 \times 10^8 \ \text{m/s}$$

a. When a platinum electrode is illuminated with light of wavelength 150 nm, the stopping potential is 2 V. What is the work function of platinum in eV?

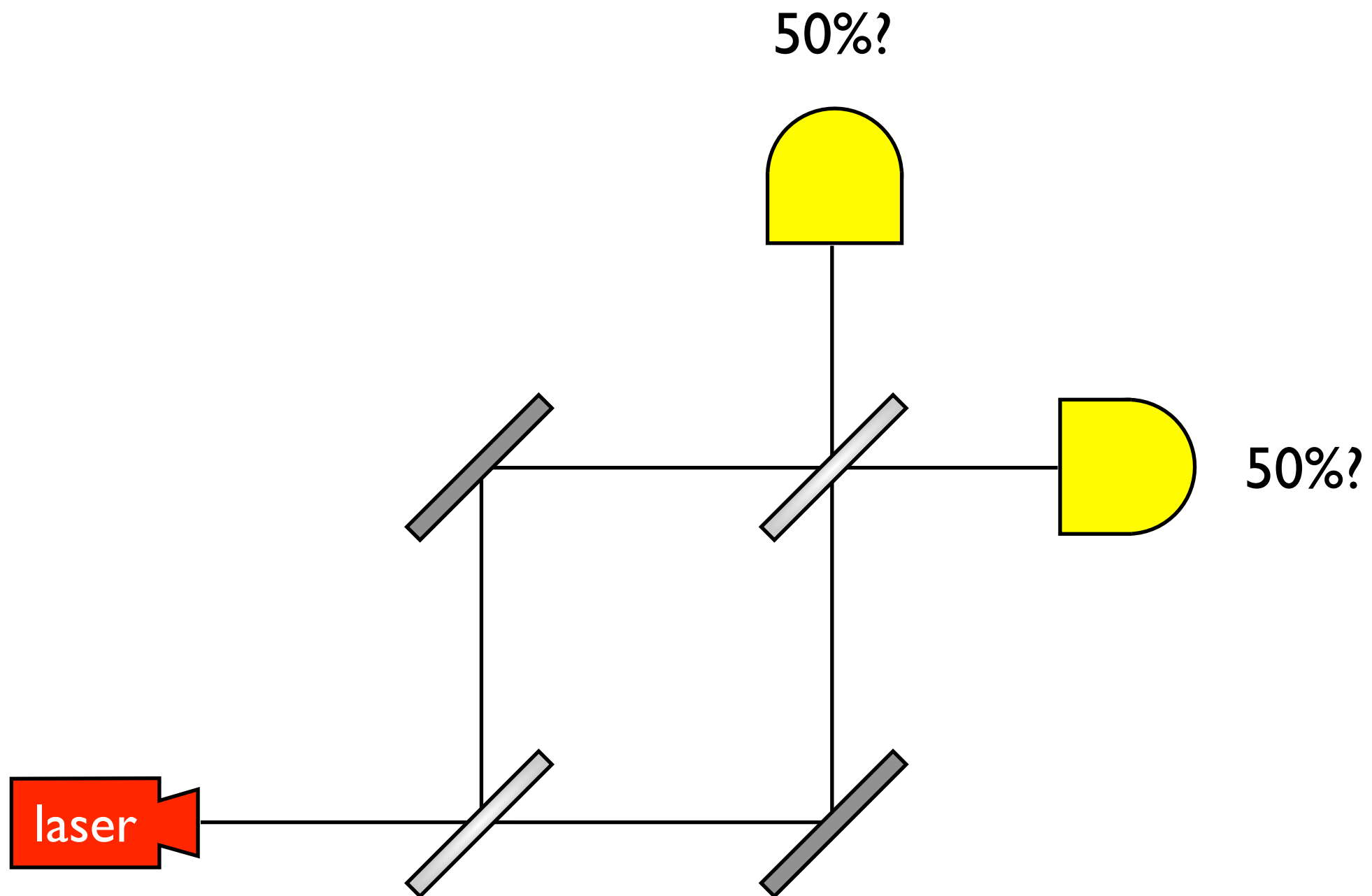b. What is the maximum wavelength of light that will eject electrons from platinum?
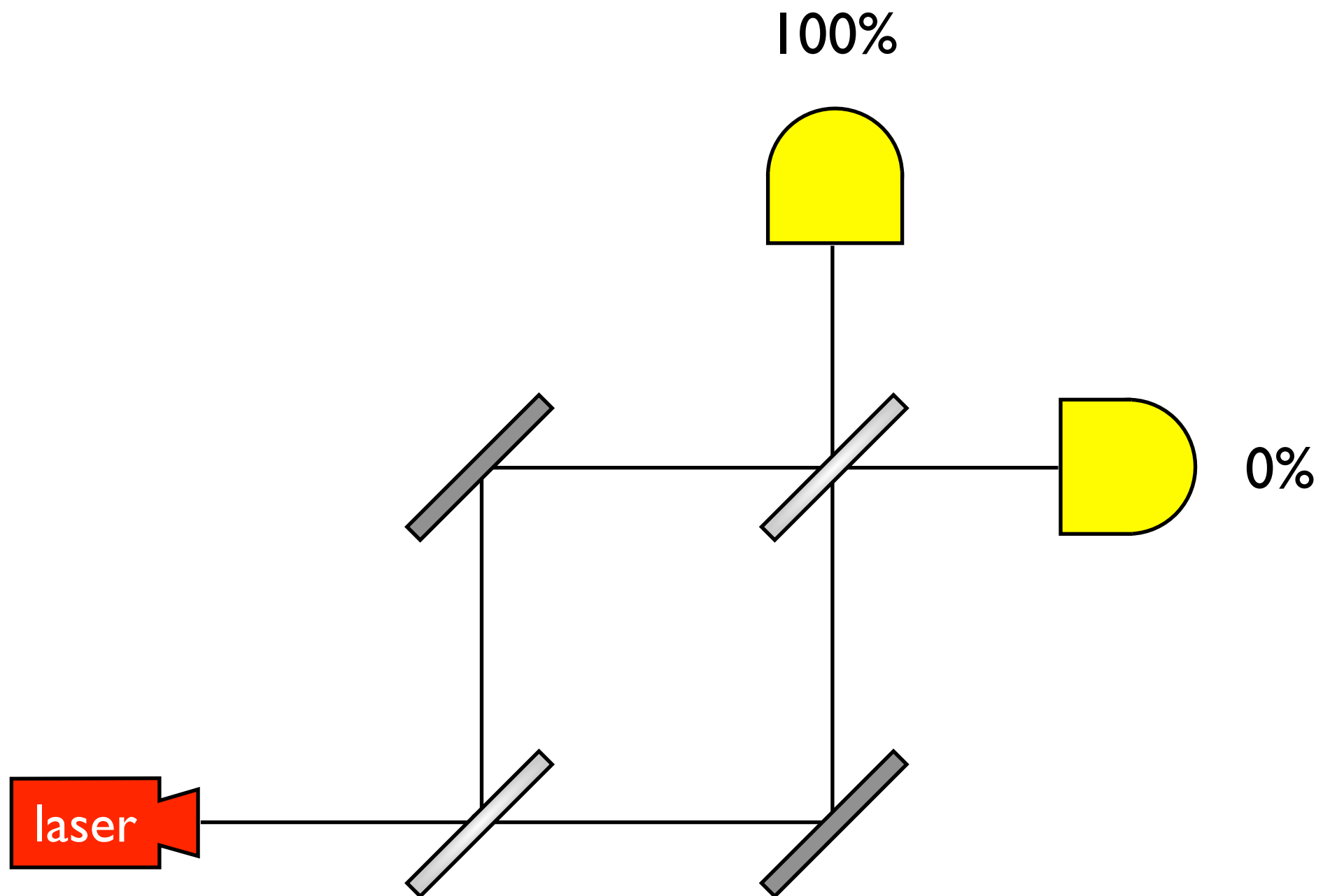
Demo

Mach-Zehnder interferometer

A simple experiment

50%

laser

detectors

50%

50/50 beamsplitter (half-silvered mirror)

Interferometer

50%?

50%?

laser

Interferometer

100%

0%

laser

Mathematical model

project onto $|0\rangle$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

project onto $|1\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

laser

$H = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
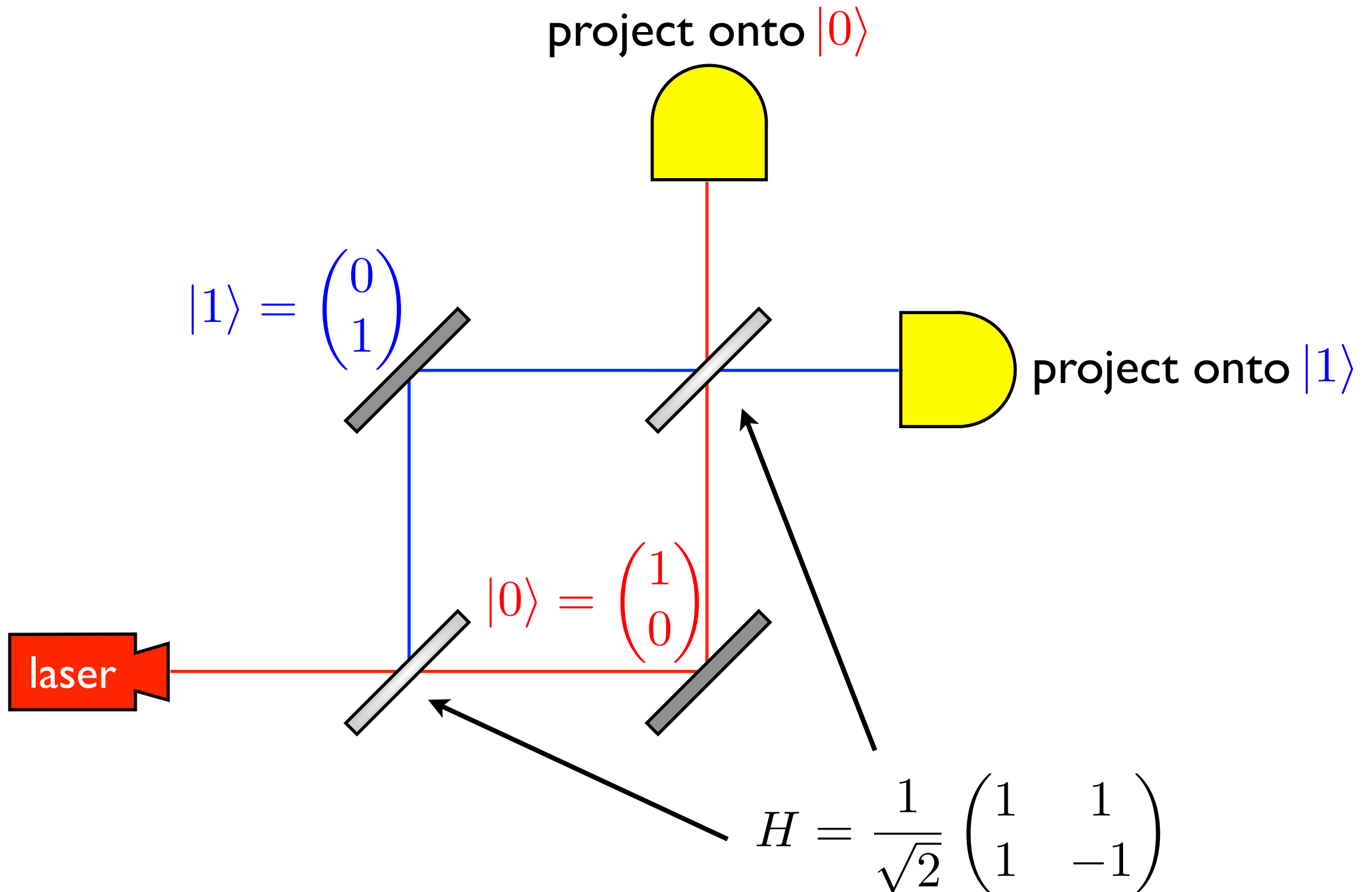
Calculation
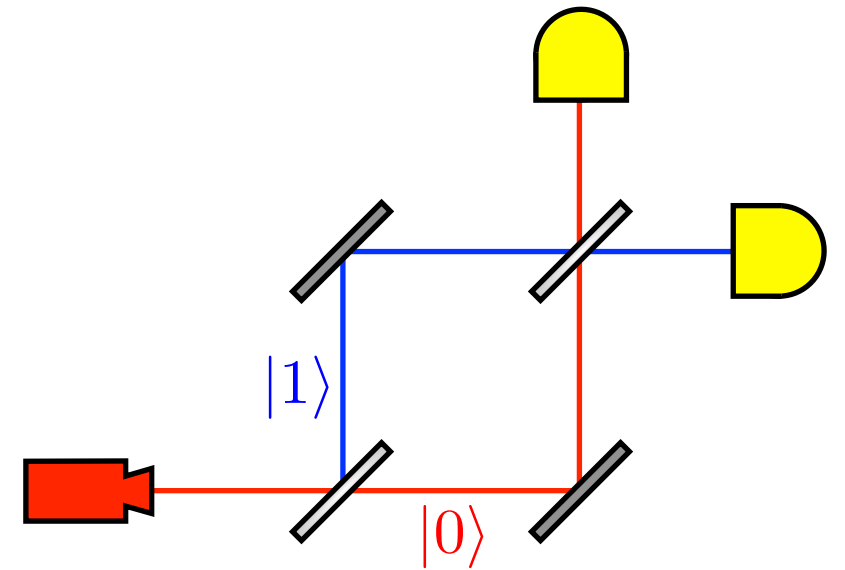
Initial state: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

After first beamsplitter:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Probability of measuring $|0\rangle$:

$$|\langle 0|0\rangle|^2 = 1$$

After second beamsplitter:

$$H\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
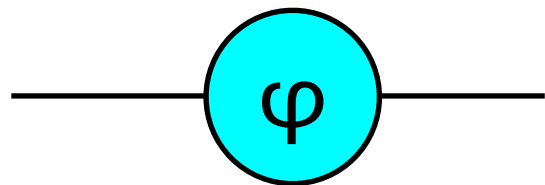
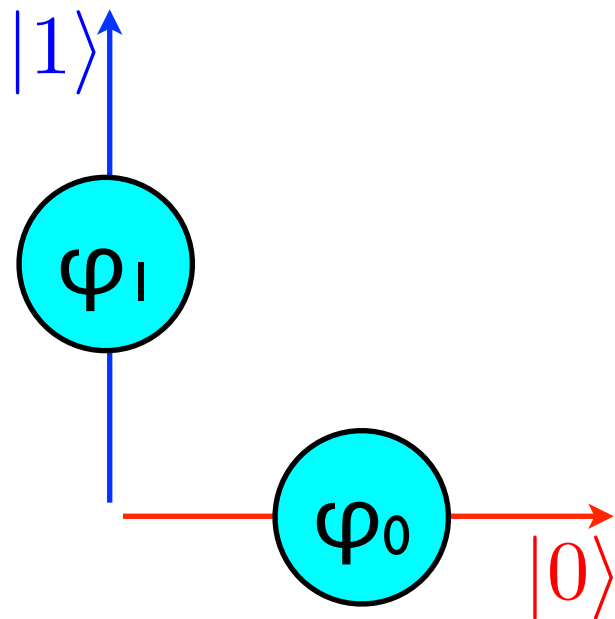Probability of measuring $|1\rangle$:

$$|\langle 1|0\rangle|^2 = 0$$

Phase shifter

Another simple optical element is a phase shifter, which shifts the phase of the light passing through it by some amount.
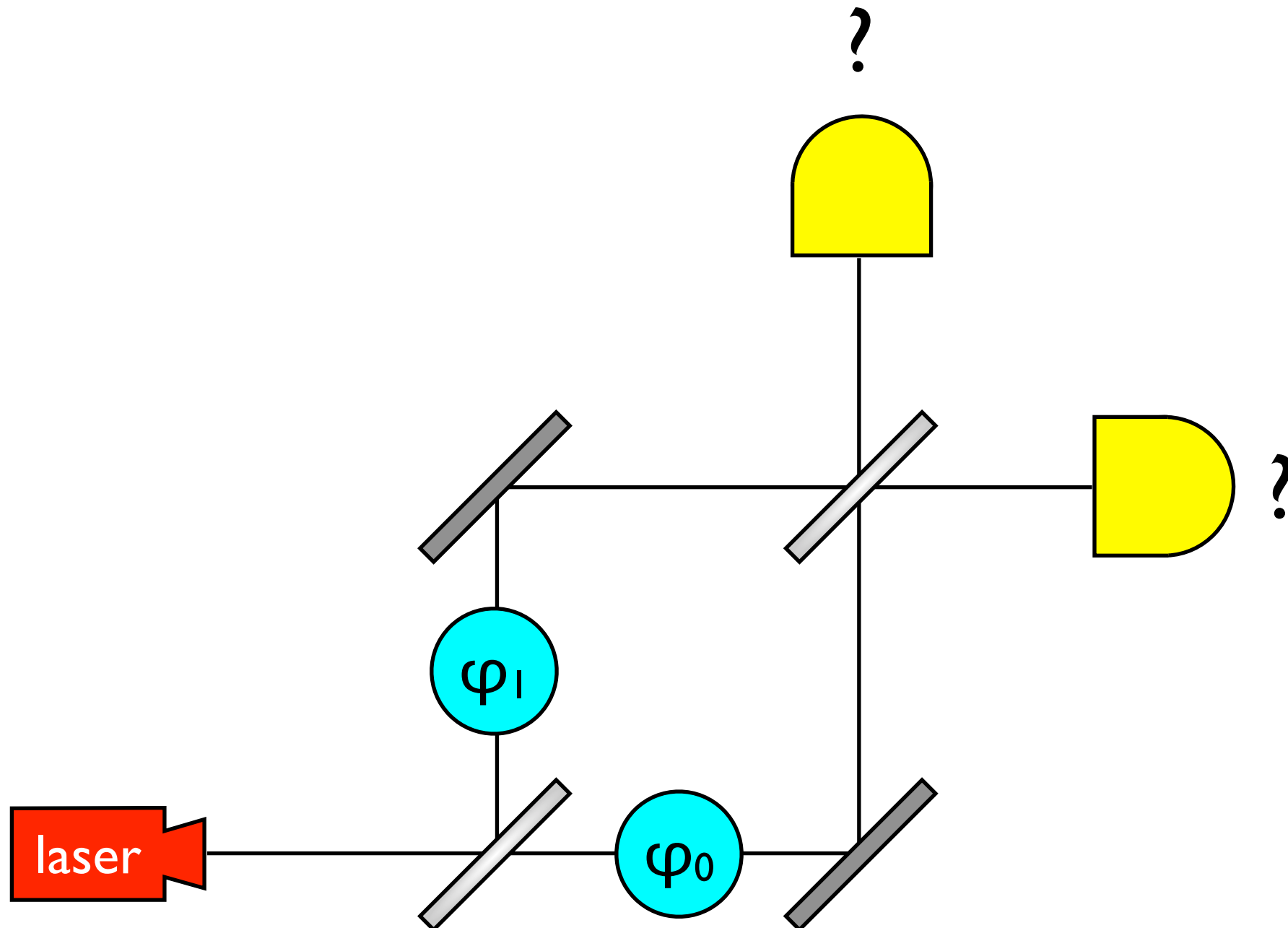

multiplies this portion of the state by $e^{i\varphi}$

implements the unitary transformation

$$\begin{pmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix}$$

Exercise: Interferometry with phase shifts

Deutsch's problem

Given: A function $f: \{0,1\} \rightarrow \{0,1\}$

(As a black box: You can call the function f, but you can't read its source code.)

Task: Determine whether f is constant.

Four possible functions:

x	$f_1(x)$	x	$f_2(x)$
0	0	0	1
1	0	1	1

constant

x	$f_3(x)$	x	$f_4(x)$
0	0	0	1
1	1	1	0

not constant
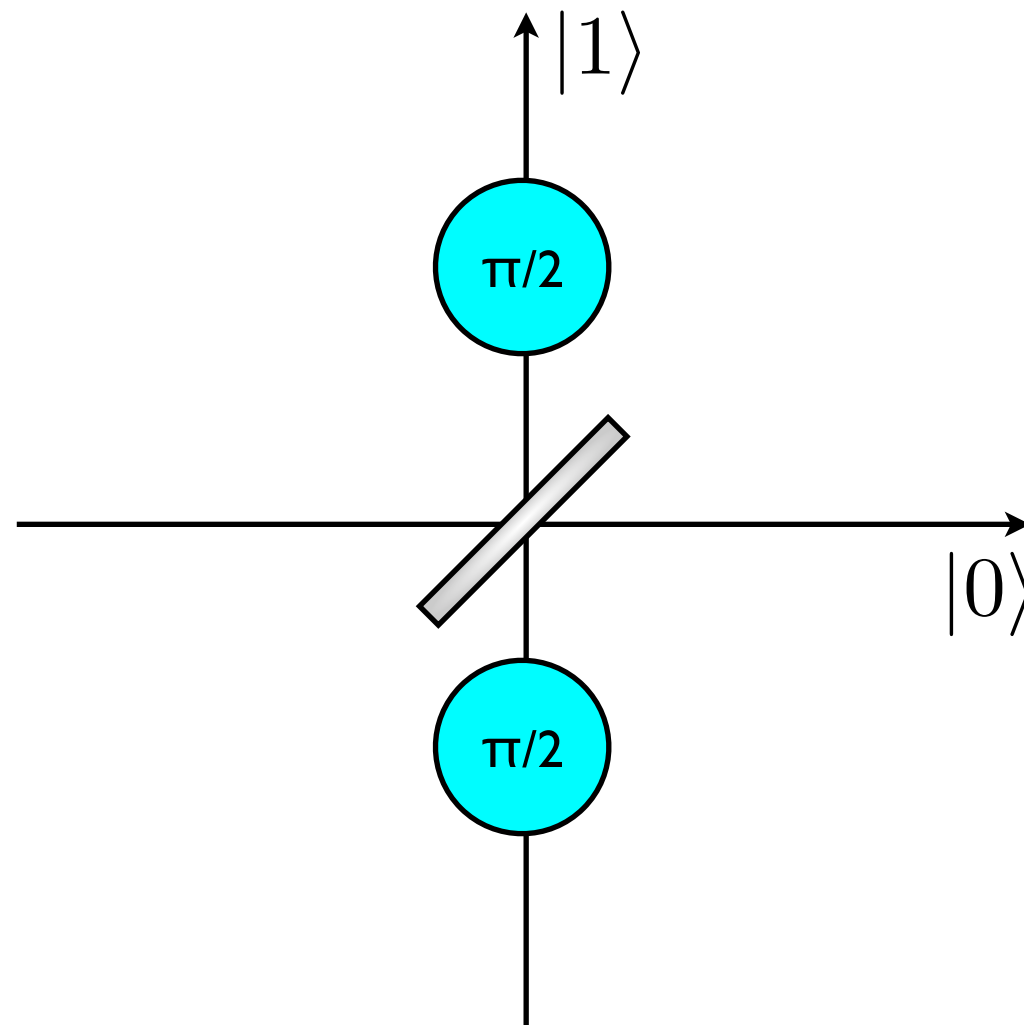
Classically, two function calls are required to solve this problem.

Deutsch's algorithm as interferometry

f is constant

f is not constant

π f(1)

π f(0)

laser
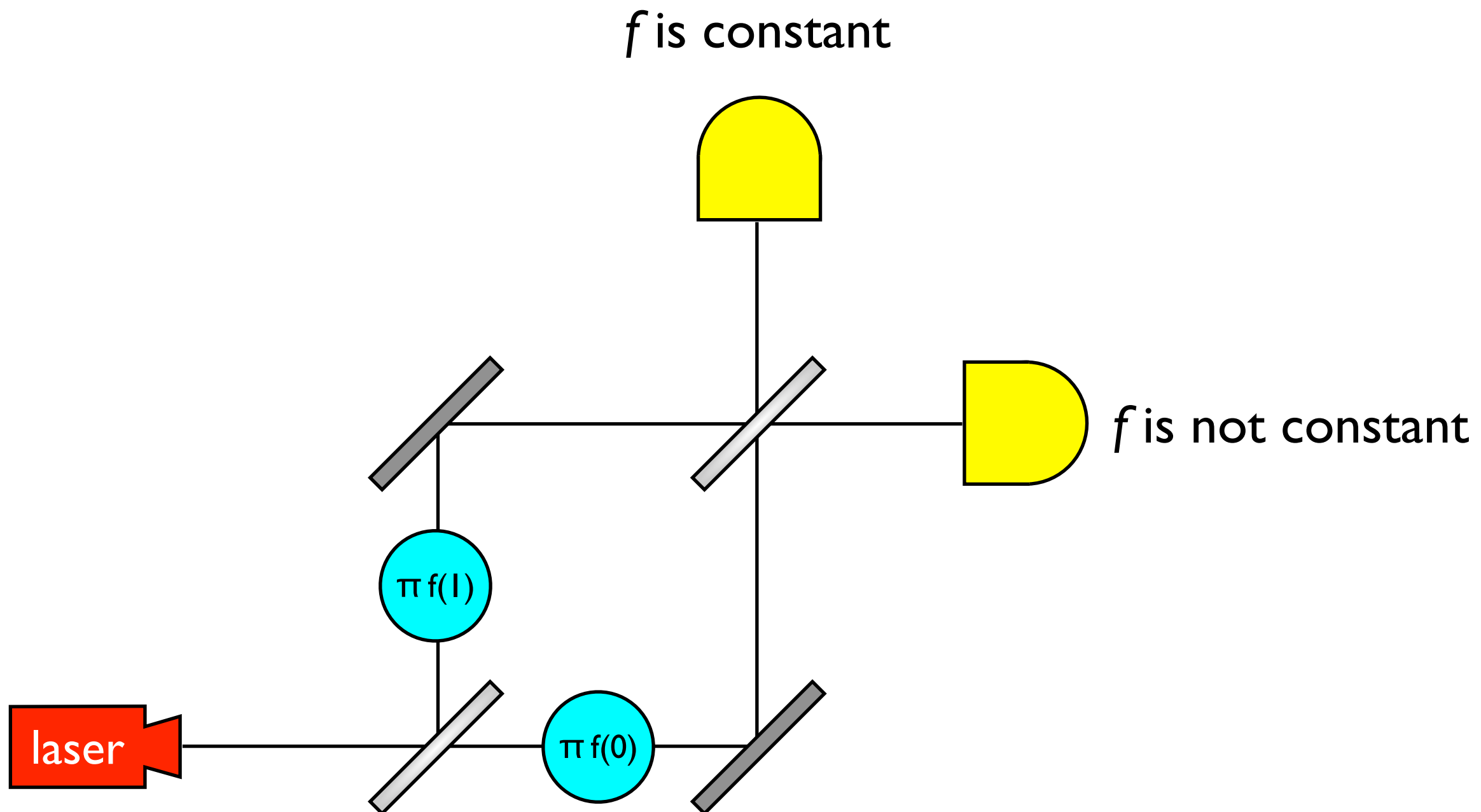
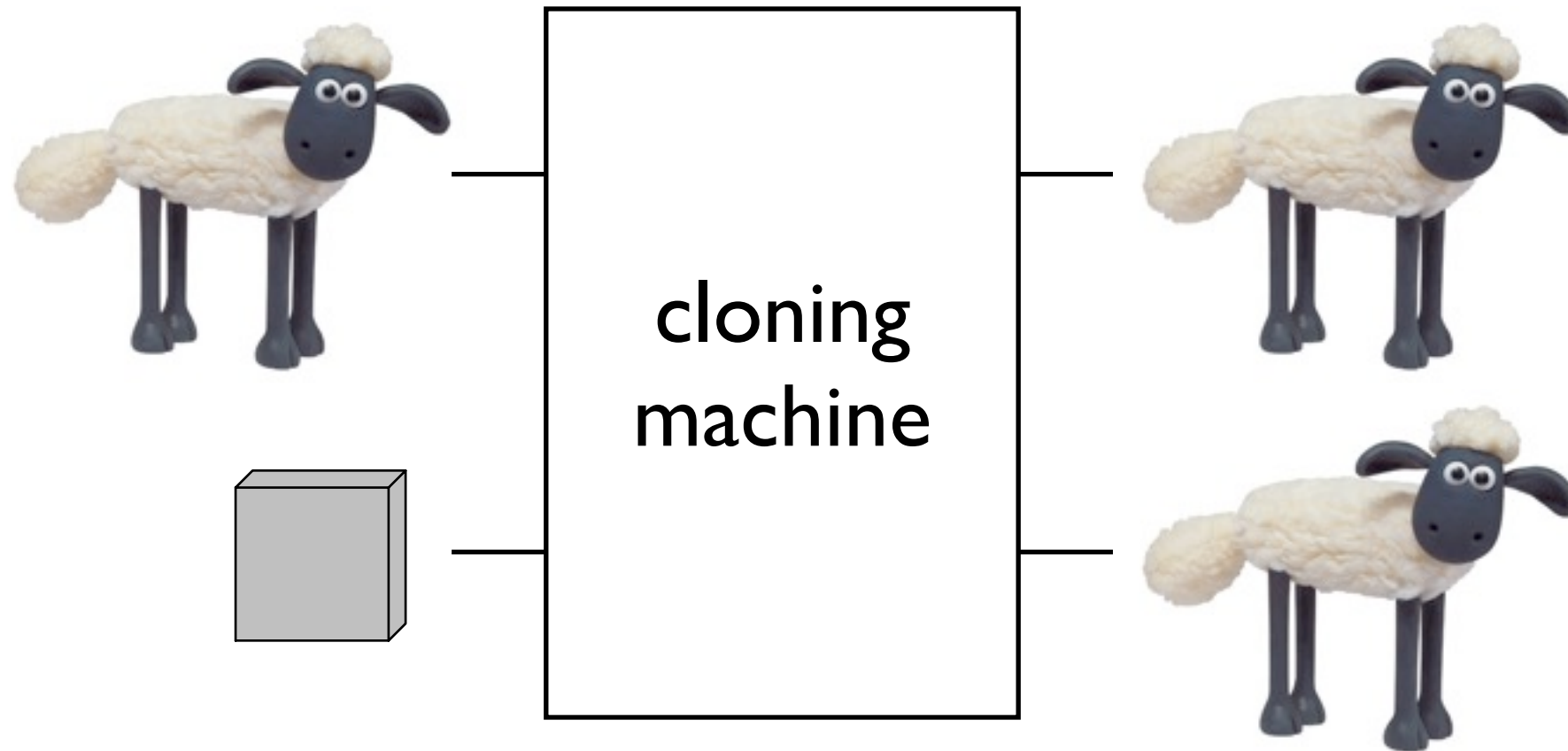# Exercise: More linear optics

What unitary transformation is implemented by the following optical setup?

No-cloning theorem

Classical cloning

In principle, such a device is possible.

Classical cloning (digital)

file1.txt

Lorem ipsum dolor sit amet, consectetur adipisicing elit...

file2.txt

cloning machine
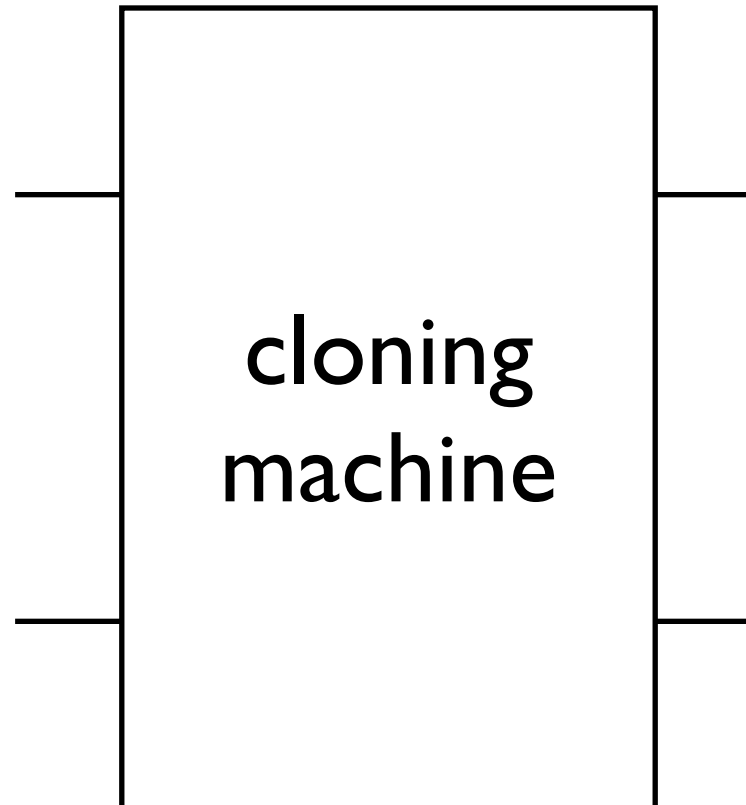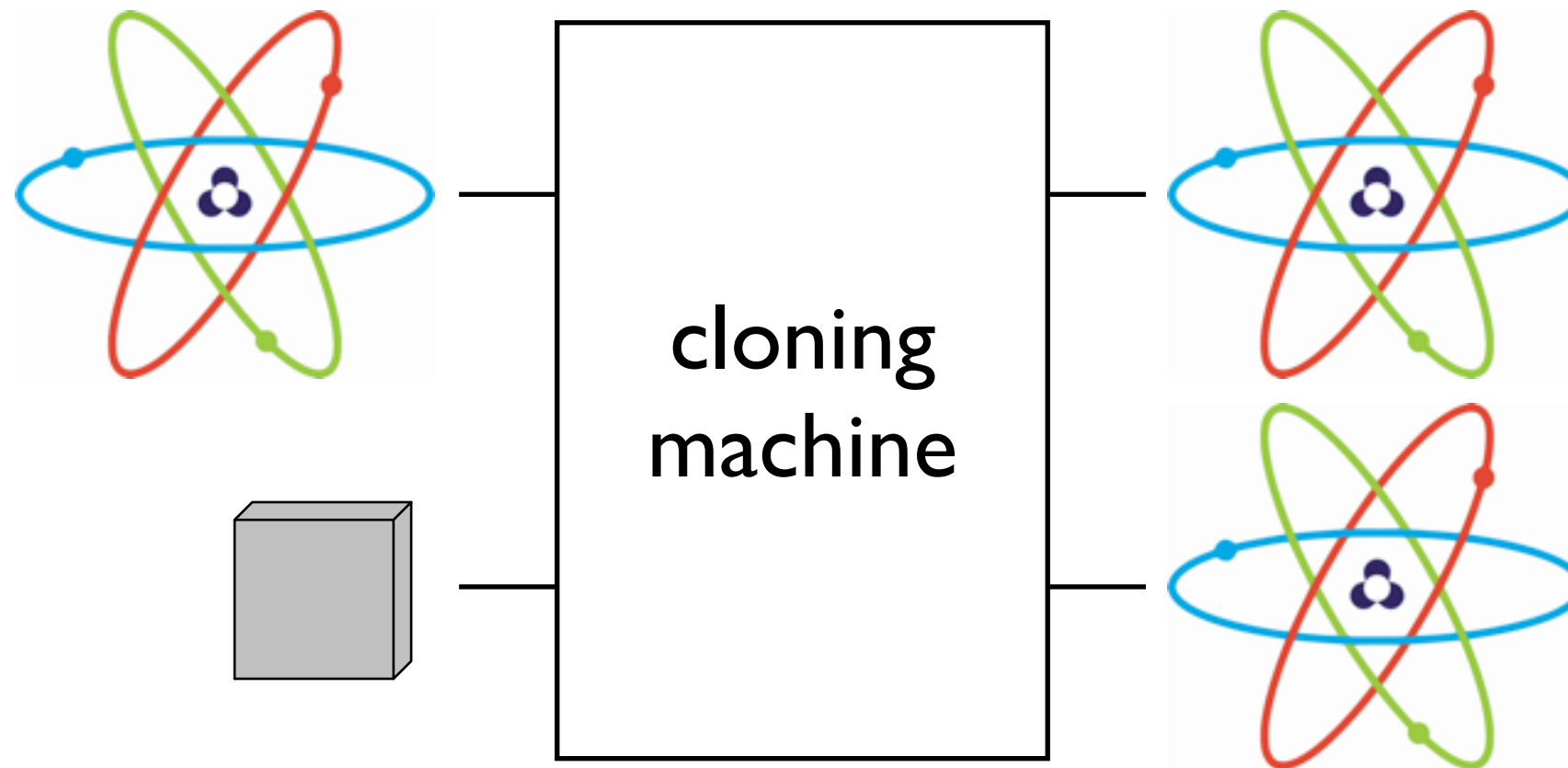
file1.txt

Lorem ipsum dolor sit amet, consectetur adipisicing elit...

file2.txt

Lorem ipsum dolor sit amet, consectetur adipisicing elit...

Quantum cloning?

The uncertainty principle prevents us from learning an unknown quantum state.

Such a device is impossible!

Quantum cloning?

$$\alpha|0\rangle + \beta|1\rangle \quad \text{——} \quad \boxed{\begin{array}{c} \text{cloning} \\ \text{machine} \end{array}} \quad \text{——} \quad \alpha|0\rangle + \beta|1\rangle$$

$$|0\rangle \quad \text{——} \quad \alpha|0\rangle + \beta|1\rangle$$
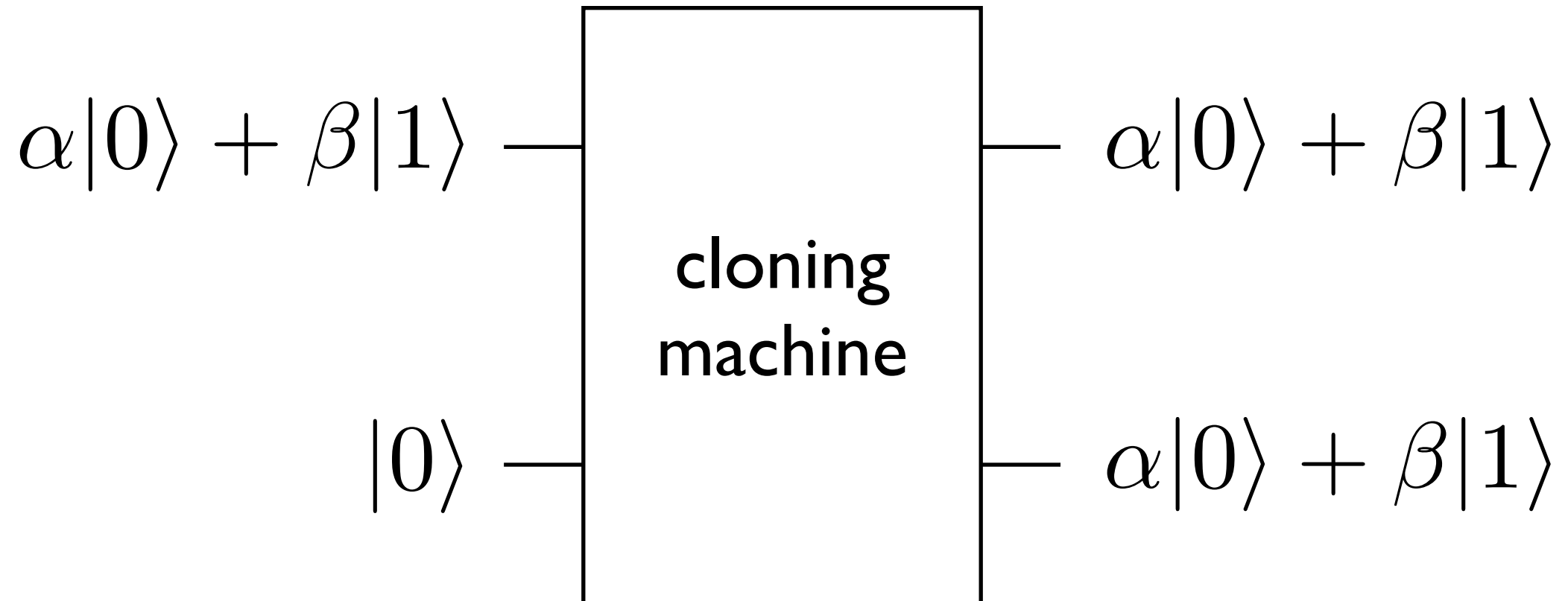
This is also impossible.

Even digital quantum information (qubits) cannot be cloned.

No-cloning theorem

Theorem [Wootters, Zurek, Dieks 1982]: There is no valid quantum process that takes as input an unknown quantum state $|\psi\rangle$ and an ancillary system in a known state, and outputs two copies of $|\psi\rangle$.
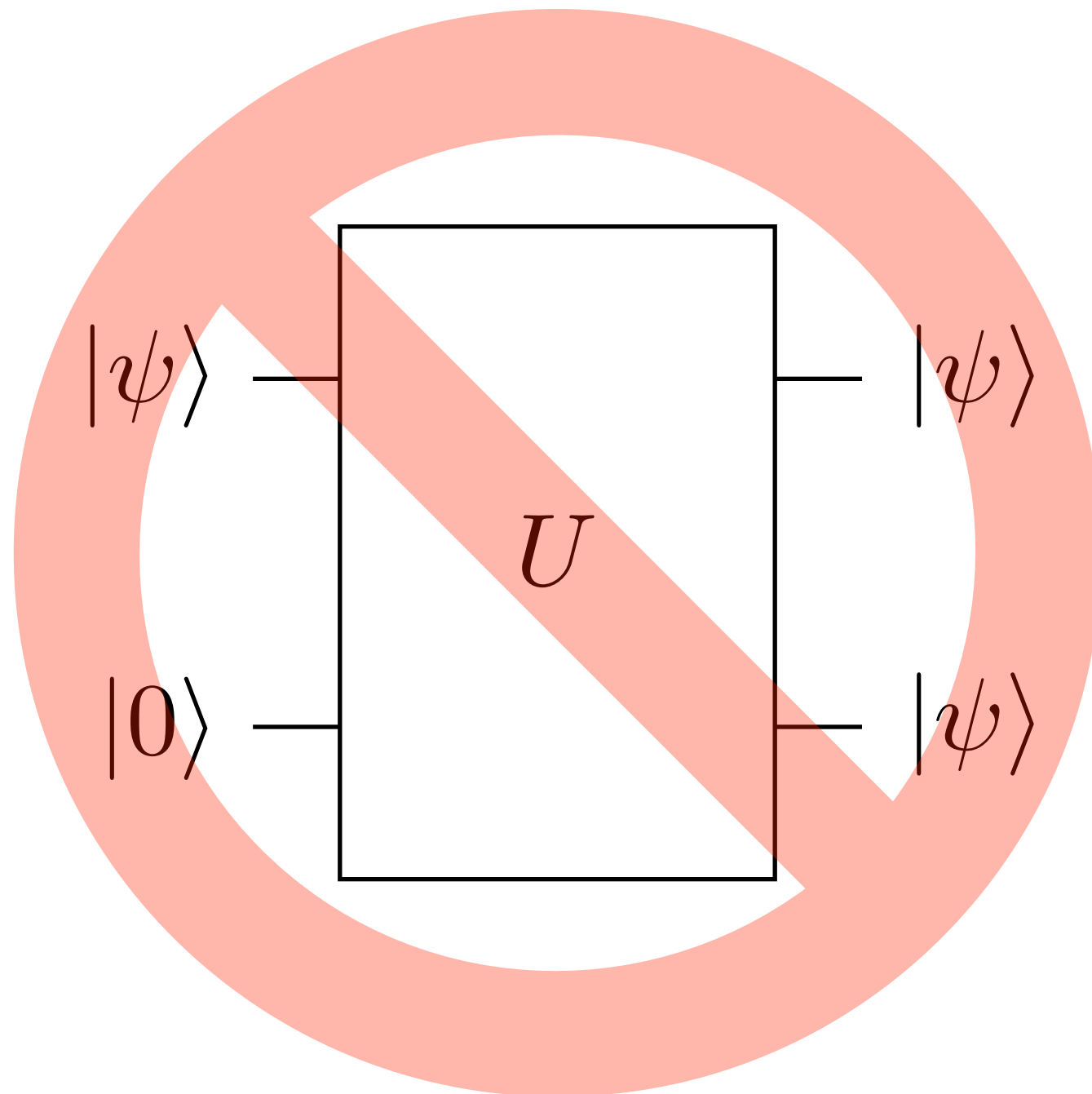
Proof of the no-cloning theorem

Consider two orthogonal states $|\psi\rangle, |\phi\rangle$. By the definition of cloning,

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

$$U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$$

By linearity,

$$U[(\alpha|\psi\rangle + \beta|\phi\rangle) \otimes |0\rangle] = \alpha U(|\psi\rangle \otimes |0\rangle) + \beta U(|\phi\rangle \otimes |0\rangle)$$

$$= \alpha|\psi\rangle \otimes |\psi\rangle + \beta|\phi\rangle \otimes |\phi\rangle$$

But again by the definition of cloning, we should have

$$U[(\alpha|\psi\rangle + \beta|\phi\rangle) \otimes |0\rangle]$$

$$= (\alpha|\psi\rangle + \beta|\phi\rangle) \otimes (\alpha|\psi\rangle + \beta|\phi\rangle)$$

$$= \alpha^2|\psi\rangle \otimes |\psi\rangle + \alpha\beta|\psi\rangle \otimes |\phi\rangle + \alpha\beta|\phi\rangle \otimes |\psi\rangle + \beta^2|\phi\rangle \otimes |\phi\rangle$$
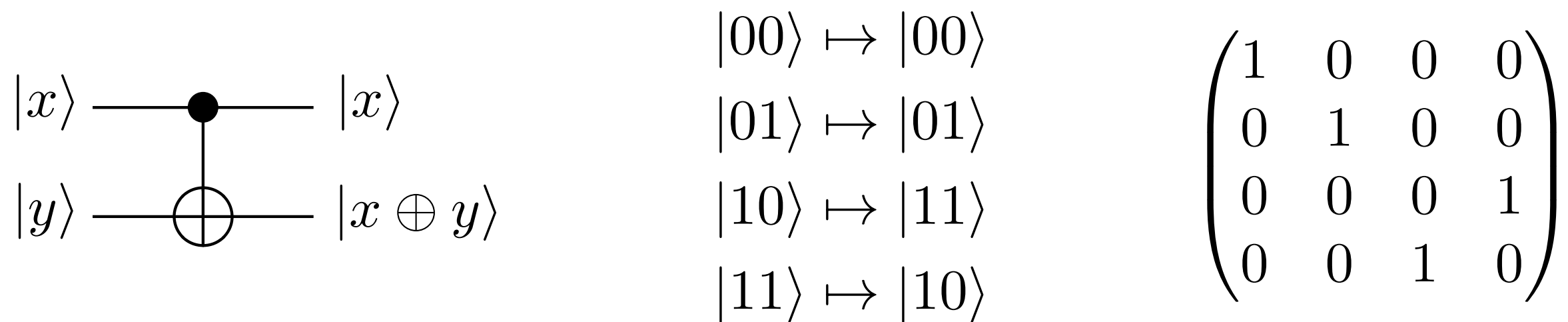
Therefore
$$\alpha^2 = \alpha, \quad \alpha\beta = 0, \quad \beta^2 = \beta$$

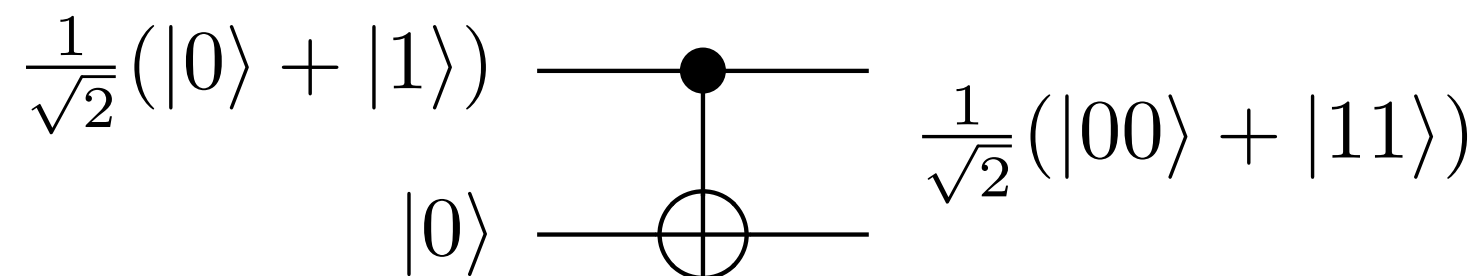So either $\alpha = 0$ or $\beta = 0$.

Cloning in a fixed basis

While we cannot copy quantum information, we can copy classical information. In particular, we can copy quantum states *in a fixed basis*.

Example: Controlled-not gate

$$|00\rangle \mapsto |00\rangle$$
$$|01\rangle \mapsto |01\rangle$$
$$|10\rangle \mapsto |11\rangle$$
$$|11\rangle \mapsto |10\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Inputting non-basis states produces an *entangled state*:

Exercise: Distinguishing non-orthogonal states

No device can clone two non-orthogonal states, and in particular, it is not possible to perfectly distinguish such states. But if we want to distinguish them, how well can we do?

Suppose Alice prepares the state $|0\rangle$ or $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, each with probability $\frac{1}{2}$.

a. If you measure in the basis $\{|0\rangle, |1\rangle\}$, with what probability can you correctly guess which state Alice prepared?

b. What if you measure in the basis $\{|+\rangle, |-\rangle\}$, where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$?

c. Can you think of another measurement that distinguishes the states with higher probability? (Hint: Consider the given states as polarizations of light. How would you orient a polarizer to get the most information about which polarization was prepared?)