Optimal quantum adversary lower bounds for ordered search

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Waterloo

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Rutgers

arXiv:0708.3396

Quantum speedup

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Constant:	Parity	n	$n/2$

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Problem: Compute a function $f : S \to \Sigma$

Input set: $S \subseteq \{0,1\}^n$ Output set: Σ

Fix some (unknown) input $x \in S$. Given a black box for the bits of x, how many queries are required to compute $f(x)$?

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(Deterministic) query complexity: n

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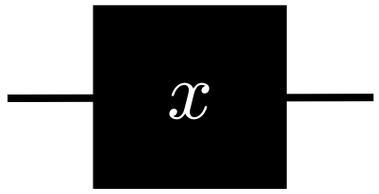

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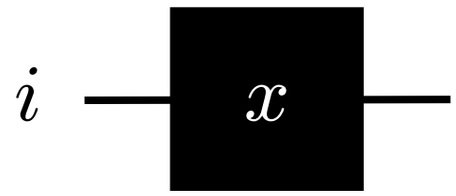

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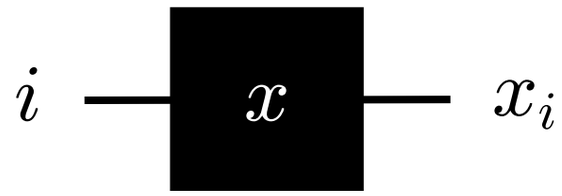

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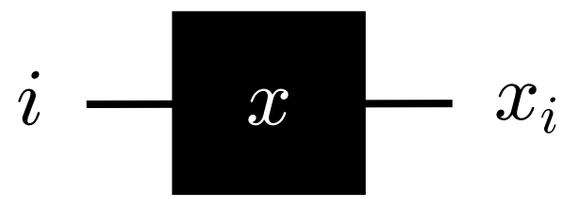

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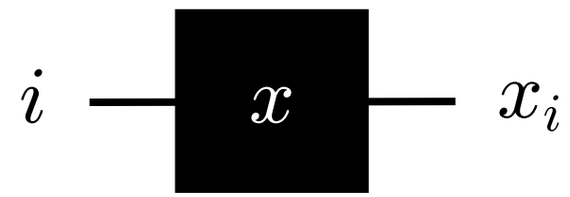

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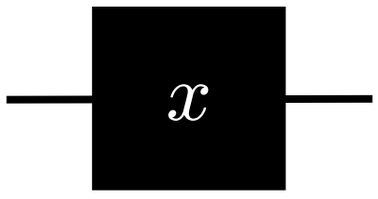

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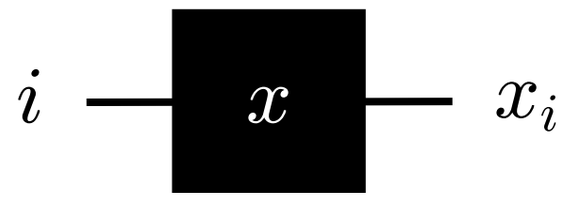

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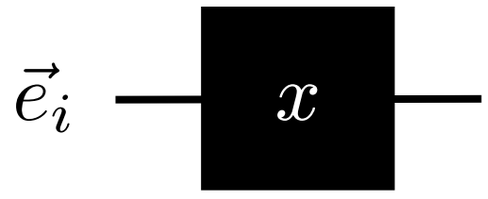

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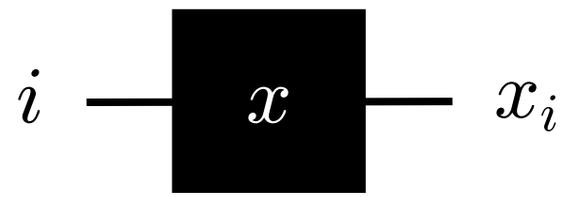

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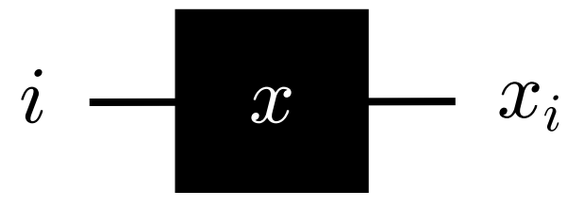

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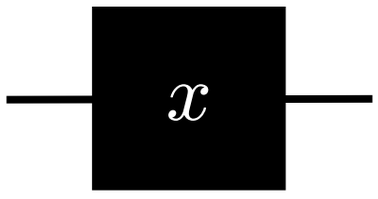

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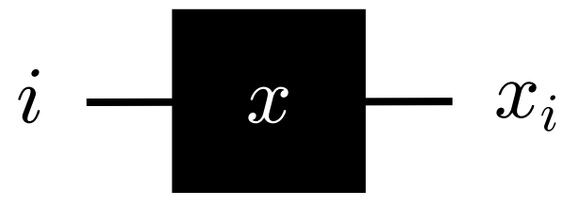

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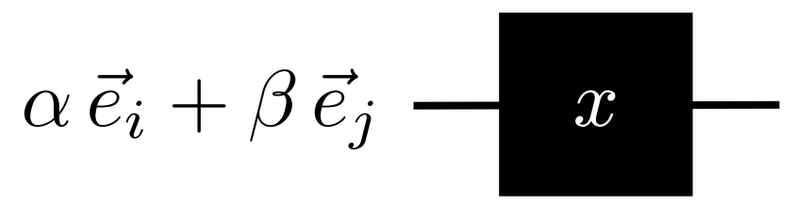

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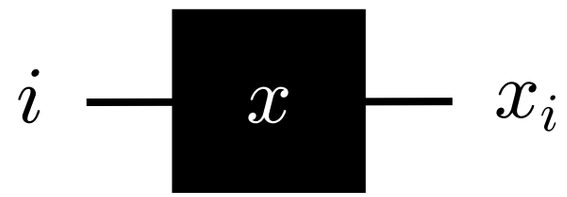

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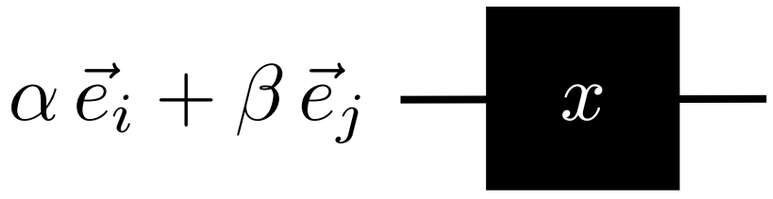

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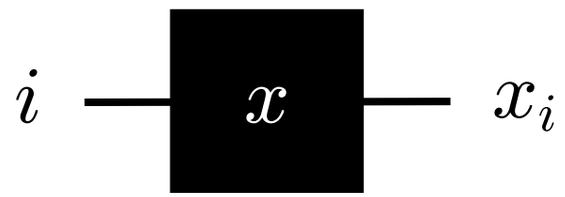

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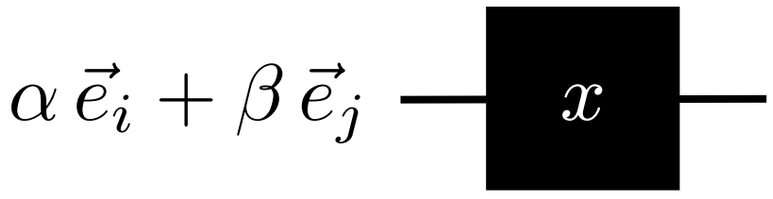

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Quantum query complexity: Minimum number of quantum queries required to quantum compute $f(x)$, given a black box for the input x.

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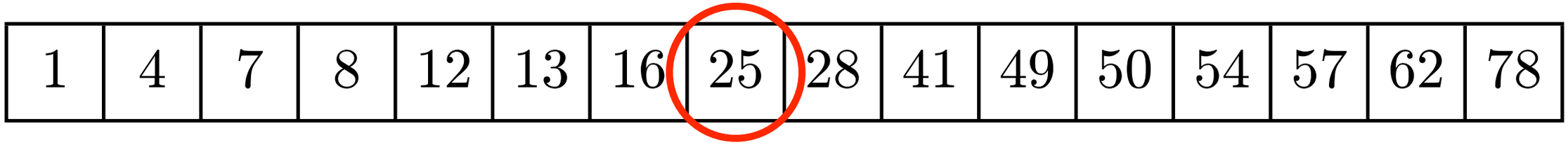

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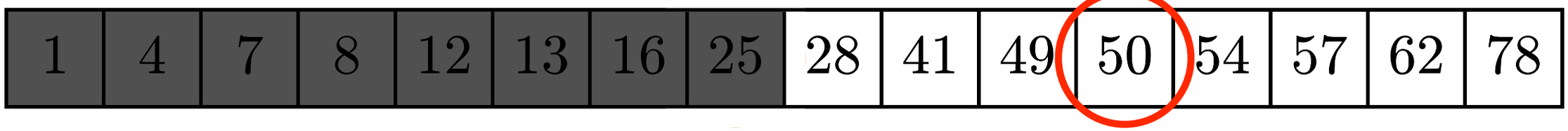

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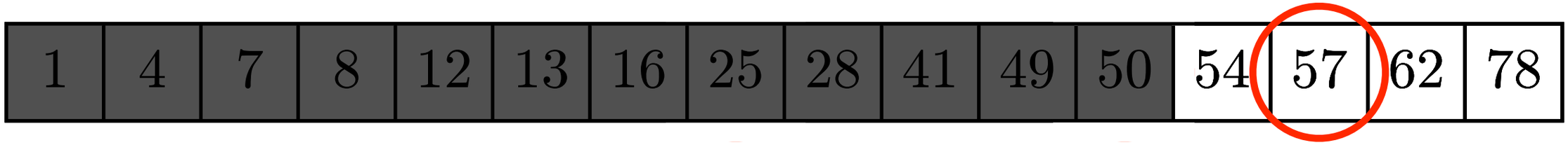

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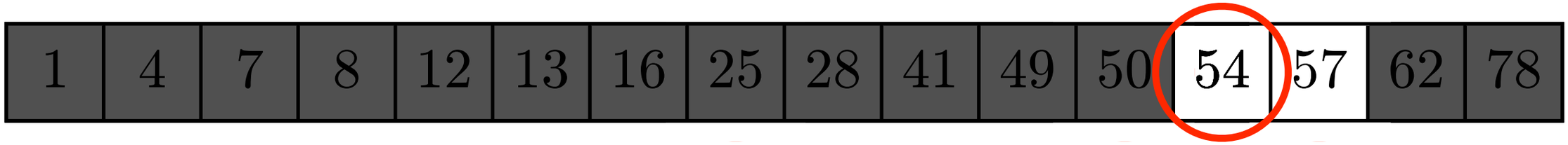

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In the above example, we have $x =$

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Høyer, Neerbek, Shi 01

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Quantum query complexity: $c \log_2 n$ for some c. What is c?

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Theorem.
This is asymptotically optimal among all adversary lower bounds.

Quantum query complexity: $c \log_2 n$ for some c. What is c?

The quantum adversary method

$$\mathrm{ADV}(f) := \max_{\substack{\Gamma \geq 0 \\ \Gamma \neq 0}} \frac{\|\Gamma\|}{\max_i \|\Gamma_i\|}$$

where Γ is an $|S| \times |S|$ matrix

entries $\Gamma[x, y]$ correspond to pairs of inputs $x, y \in S$

$\Gamma[x, y] = 0$ if $f(x) = f(y)$

$$\Gamma_i[x, y] := \begin{cases} 0 & x_i = y_i \\ \Gamma[x, y] & x_i \neq y_i \end{cases}$$

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Theorem [Ambainis 00]: (Q. query complexity of f) $\geq \frac{1}{2}\mathrm{ADV}(f)$.

Proof idea: Define a progress measure for algorithms. It starts at 0 and must reach $\|\Gamma\|$ for the algorithm to succeed; the maximum change per query is $2 \max_i \|\Gamma_i\|$.

Semidefinite programming and $\mathrm{ADV}(f)$

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Notice that computing $\mathrm{ADV}(f)$ is a semidefinite program!

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Definition: An *automorphism* of $f : S \to \Sigma$ is a permutation $\pi \in S_n$ with

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Automorphism principle [Høyer, Lee, Špalek 07]: If π is an automorphism of f, then we can choose an optimal adversary matrix Γ satisfying $\Gamma[x, y] = \Gamma[\pi(x), \pi(y)]$ for all pairs of inputs x, y.

Furthermore, if the automorphism group is transitive, then the uniform vector is a principal eigenvector of Γ, and all $\|\Gamma_i\|$ are equal.

Symmetrizing ordered search

Recall ordered search function: e.g., for $n = 4$, the inputs are

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Farhi, Goldstone, Gutmann, Sipser 99

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Again, the problem is to identify the input ($f(x) = x$).
Now the automorphism group is cyclic ($2n$ elements).
We call this problem OSP_n.

Farhi, Goldstone, Gutmann, Sipser 99

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Asymptotically, this is negligible.

Adversary SDP for ordered search

$$\Gamma = \begin{array}{c} \begin{matrix} 11110000 & 01111000 & 00111100 & 00011110 & 00001111 & 10000111 & 11000011 & 11100001 \end{matrix} \\ \left[\begin{matrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\ \gamma_1 & 0 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_2 & \gamma_8 & 0 & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} \\ \gamma_3 & \gamma_9 & \gamma_{14} & 0 & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} \\ \gamma_4 & \gamma_{10} & \gamma_{15} & \gamma_{19} & 0 & \gamma_{23} & \gamma_{24} & \gamma_{25} \\ \gamma_5 & \gamma_{11} & \gamma_{16} & \gamma_{20} & \gamma_{23} & 0 & \gamma_{26} & \gamma_{27} \\ \gamma_6 & \gamma_{12} & \gamma_{17} & \gamma_{21} & \gamma_{24} & \gamma_{26} & 0 & \gamma_{28} \\ \gamma_7 & \gamma_{13} & \gamma_{18} & \gamma_{22} & \gamma_{25} & \gamma_{27} & \gamma_{28} & 0 \end{matrix}\right] \end{array} \begin{matrix} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{matrix}$$

Adversary SDP for ordered search

By the automorphism principle, we can assume

$$\Gamma = \begin{array}{c} \begin{matrix} 11110000 & 01111000 & 00111100 & 00011110 & 00001111 & 10000111 & 11000011 & 11100001 \end{matrix} \\ \begin{bmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 \\ \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 \end{bmatrix} \begin{matrix} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{matrix} \end{array}$$

Spectral norm achieved by uniform eigenvector: $\gamma_n + 2\sum_{i=1}^{n-1} \gamma_i$

Adversary SDP for ordered search

Also by the automorphism principle, it suffices to consider

$$\Gamma_8 = \begin{array}{c} \begin{array}{cccccccc} 11110000 & 01111000 & 00111100 & 00011110 & 00001111 & 10000111 & 11000011 & 11100001 \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ 0 & 0 & 0 & 0 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 \\ 0 & 0 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 \\ 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \\ \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & 0 & 0 & 0 & 0 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{array} \end{array}$$

In general, $\|\Gamma_{2n}\| = \|\mathrm{Toeplitz}(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1)\|$.

Adversary SDP for ordered search

Primal:

$$\max \gamma_n + 2\sum_{i=1}^{n-1} \gamma_i \quad \text{subject to} \quad \|\text{Toeplitz}(\gamma_n, \ldots, \gamma_1)\| \leq 1,\ \gamma_i \geq 0$$

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Hilbert matrix: $\left\| \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & & \\ \frac{1}{3} & & & \\ \vdots & & & \end{bmatrix} \right\| = \pi$

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$$\min \text{tr}(P) \quad \text{subject to} \quad P \succeq 0,\ \ \text{tr}_i(P) \geq 1 \text{ for } i = 0, \ldots, n-1$$

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Theorem.

$$\text{ADV}(\text{OSP}_{2m}) = 2\sum_{i=0}^{m-1} \left(\frac{\binom{2i}{i}}{4^i} \right)^2$$

$$\text{ADV}(\text{OSP}_{2m+1}) = 2\sum_{i=0}^{m-1} \left(\frac{\binom{2i}{i}}{4^i} \right)^2 + \left(\frac{\binom{2m}{m}}{4^m} \right)^2$$

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Primal is more technical but uses similar ideas.

A binomial identity

Recall $\xi_i := \dfrac{\binom{2i}{i}}{4^i}$

Proposition. For any j, $\displaystyle\sum_{i=0}^{j} \xi_i \, \xi_{j-i} = 1.$

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GF for LHS: $\quad \dfrac{1}{1-z} = \displaystyle\sum_{i=0}^{\infty} z^i$

Asymptotic analysis

$$\mathrm{ADV}(\mathrm{OSP}_n) = 2 \sum_{i=0}^{\frac{n}{2}-1} \left(\frac{\binom{2i}{i}}{4^i} \right)^2$$

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Result follows by analyzing the logarithmic singularity at $z = 1$ using Darboux's method.

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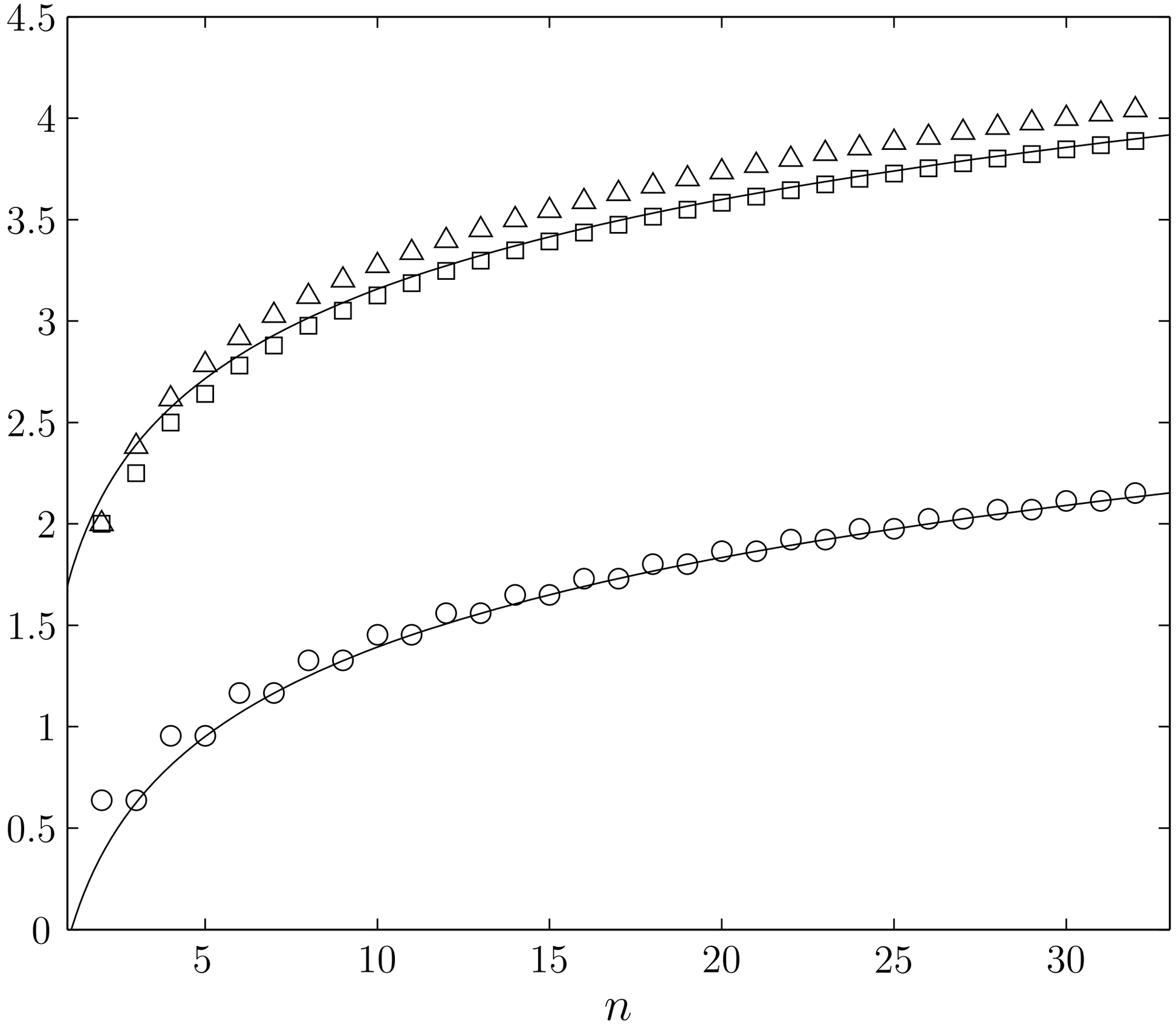
For comparison, the HNS bound says

$$\mathrm{ADV}(\mathrm{OSP}_n) \geq \frac{2}{\pi}(\ln n + \gamma - \ln 2) + O(1/n)$$


4.5
4
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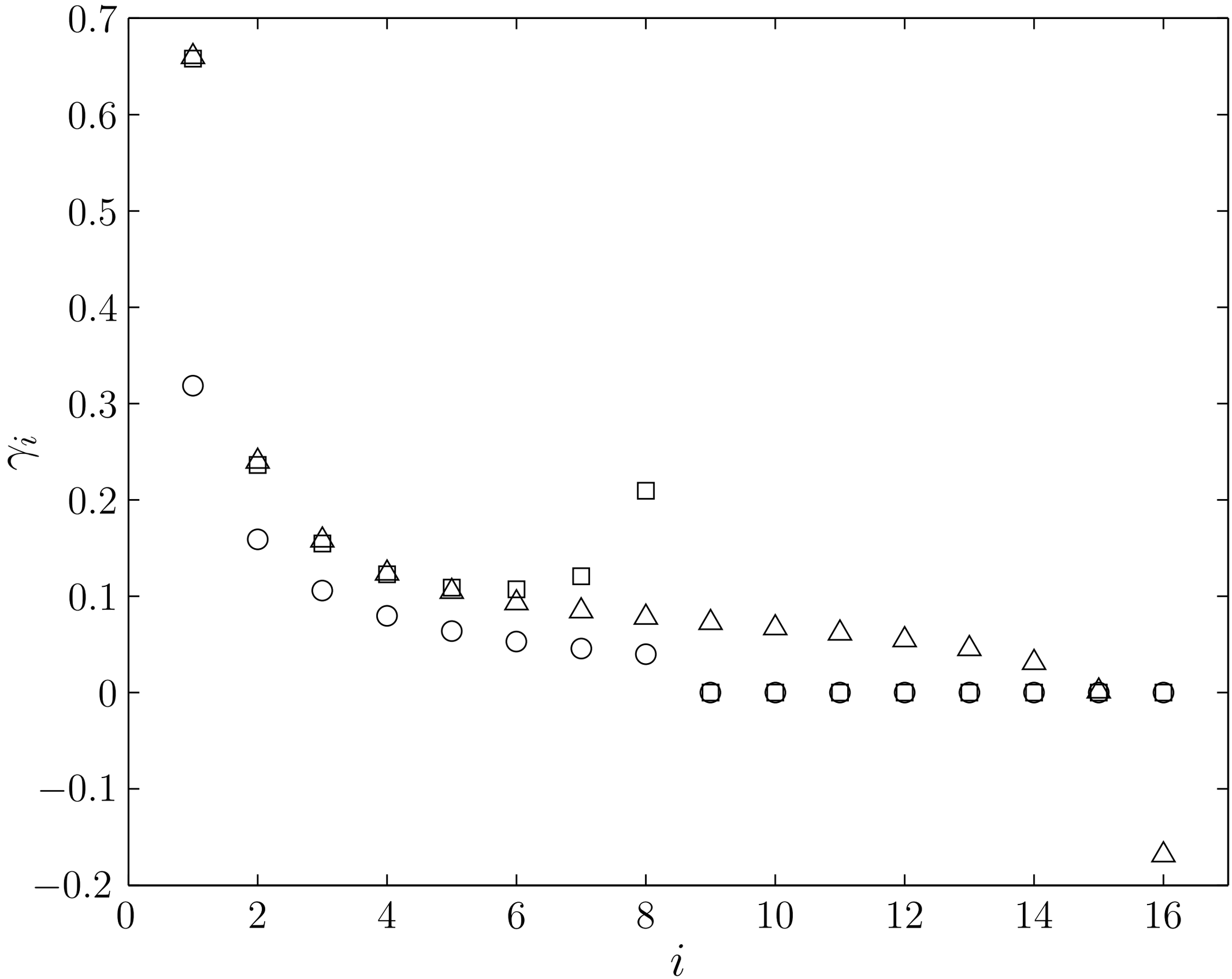
0.7
0.6
0.5
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0
−0.1
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γ_i
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8
10
12
14
16
i

The negative adversary

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Theorem [Høyer, Lee, Špalek 07]:

(Quantum query complexity of f) $\geq \frac{1}{2}\mathrm{ADV}^{\pm}(f) \geq \frac{1}{2}\mathrm{ADV}(f)$.

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Theorem [Høyer, Lee, Špalek 07]:

(Quantum query complexity of f) $\geq \frac{1}{2}\mathrm{ADV}^{\pm}(f) \geq \frac{1}{2}\mathrm{ADV}(f)$.

Furthermore, there are functions for which the negative adversary gives a significantly better lower bound.

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We give a general analysis of the spectra of such matrices.

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- Can we use insights from the optimal adversary to find a better algorithm? (Note: Quantum query complexity is an SDP.)
- Can we find optimal adversary lower bounds for other problems? (Element distinctness?)