An introduction to adiabatic quantum computation

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Outline

- Quantum computers
- Quantum computation and Hamiltonian dynamics
- The adiabatic theorem
- Adiabatic optimization
  - Examples of success
  - Example of failure
- Robustness of adiabatic QC
  - Unitary control error
  - Thermal noise

Why quantum computation?

Quantum computers can solve certain problems dramatically faster than classical computers can.

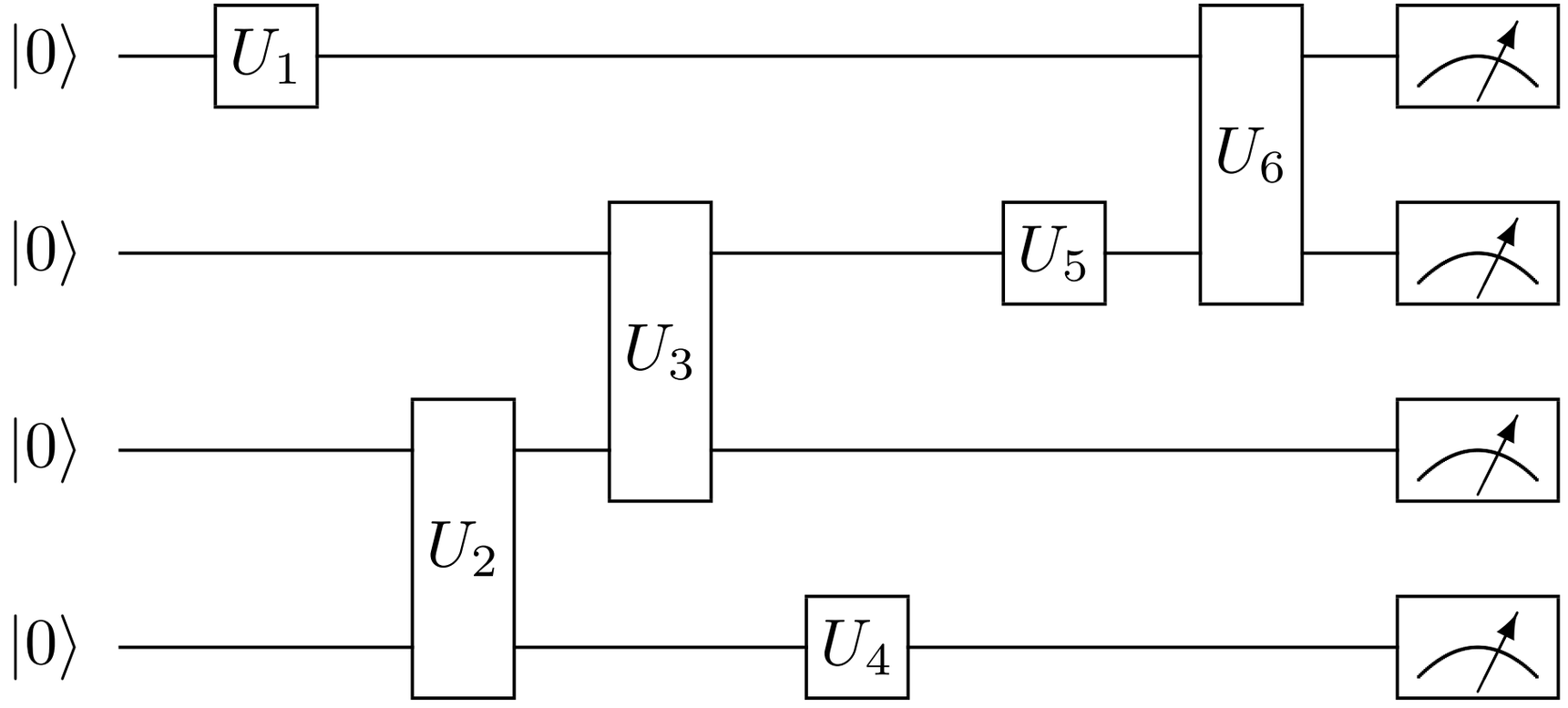
- Simulating quantum dynamics
- Factoring
- Discrete log
- Pell's equation
- Abelian HSP
- Some nonabelian HSPs
- Shifted Legendre symbol/ polynomial reconstruction
- Estimating Gauss sums
- Graph traversal
- Approximating Jones polynomial
- Counting solutions of finite field equations

Main questions:

- What other problems can we solve faster with a quantum computer?
- How can we actually build a quantum computer, despite the extreme sensitivity of quantum systems to noise?

Quantum circuits

- Prepare *n* qubits in the state $|0\cdots 0\rangle$
- Apply a sequence of poly(*n*) unitary operations acting on one or two qubits at a time
- Measure in the computational basis to get the result



Hamiltonian dynamics

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

In the circuit model, we say a unitary operation can be implemented efficiently if it can be realized (approximately) by a short sequence of one- and two-qubit gates.

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- Hamiltonians we can efficiently simulate using quantum circuits

Simulating Hamiltonian dynamics

Definition. A Hamiltonian H acting on n qubits can be *efficiently simulated* if for any error $\epsilon>0$ and time $t>0$ there is a quantum circuit U consisting of poly(n, t, $1/\epsilon$) gates such that $\| U - e^{-iHt} \| < \epsilon$.

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Basic idea: Lie product formula

$$e^{-i(H_1+\cdots+H_k)t} = (e^{-iH_1 t/r} \cdots e^{-iH_k t/r})^r + O(kt^2 \max\{\|H_j\|^2\}/r)$$

Sparse Hamiltonians

Theorem. Suppose that for any fixed *a*, we can efficiently compute all the nonzero values of $\langle a|H|b\rangle$. (In particular, there must be only polynomially many such values.) Then *H* can be simulated efficiently. [Aharonov & Ta-Shma 2003, Childs et al. 2003, Ahokas et al. 2005]

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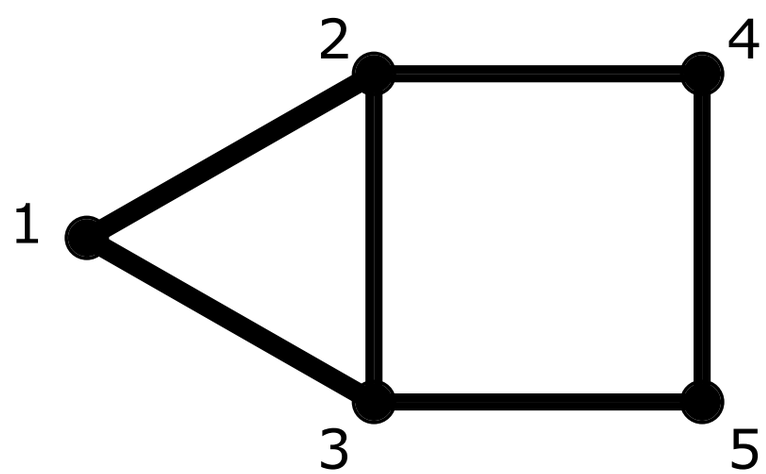
$$H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



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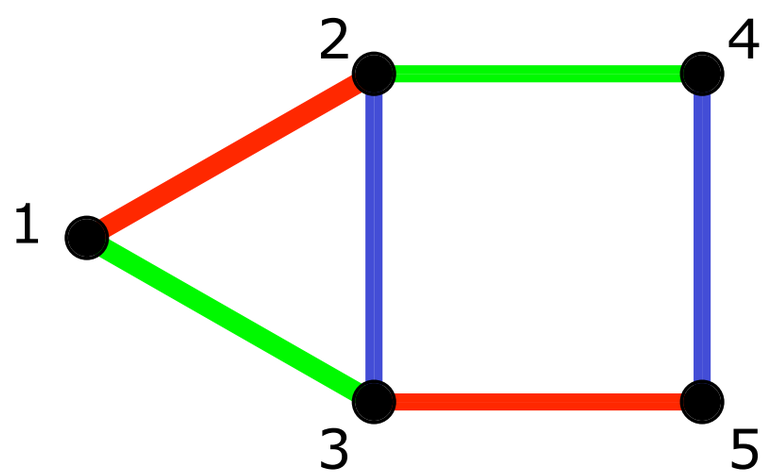

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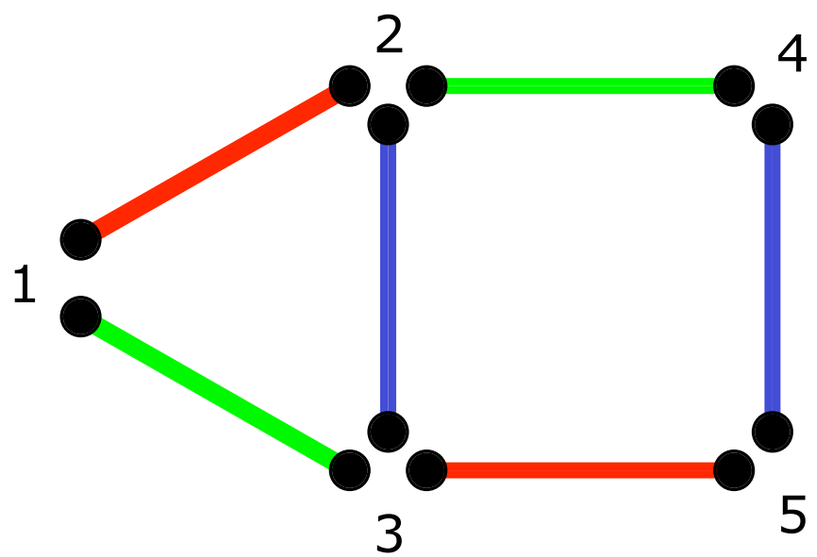
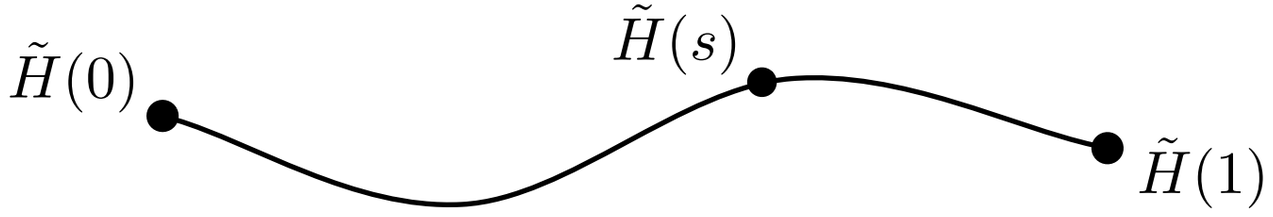


The adiabatic theorem

Let $\tilde{H}(s)$ be a smoothly varying Hamiltonian for $s \in [0,1]$



$$\tilde{H}(s) = \sum_{j=0}^{D-1} E_j(s)|E_j(s)\rangle\langle E_j(s)|$$

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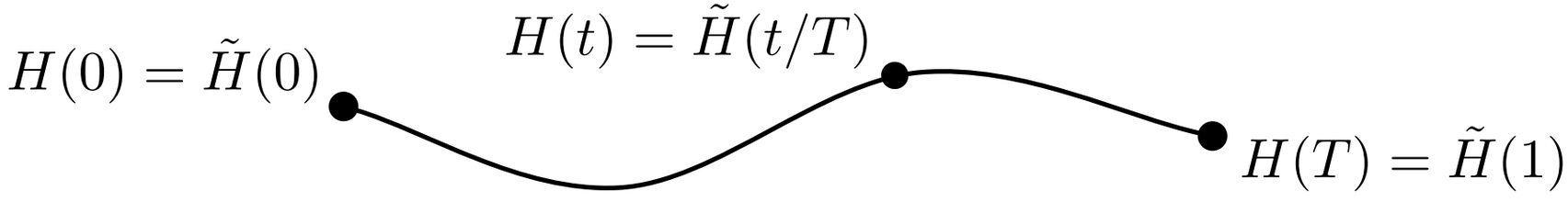

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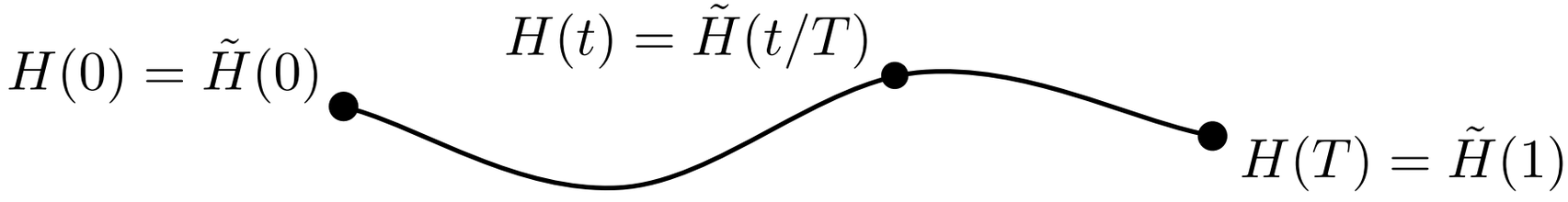

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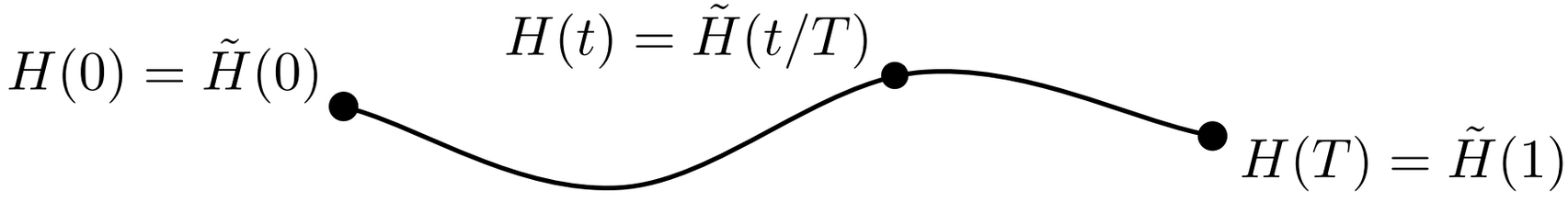

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For large T, $|\psi(T)\rangle \approx |E_0(1)\rangle$. But how large must it be?

Approximately adiabatic evolution

The total run time required for adiabaticity depends on the spectrum of the Hamiltonian.

Gap: $\Delta(s) = E_1(s) - E_0(s)\,, \quad \Delta = \min_{s\in[0,1]} \Delta(s)$

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Theorem. [Teufel 2003 + perturbation theory]

$$T \geq \frac{4}{\epsilon}\left[\frac{\|\dot{\tilde{H}}(0)\|}{\Delta(0)^2} + \frac{\|\dot{\tilde{H}}(1)\|}{\Delta(1)^2} + \int_0^1 \mathrm{d}s \left(10\frac{\|\dot{\tilde{H}}\|^2}{\Delta^3} + \frac{\|\ddot{\tilde{H}}\|}{\Delta}\right)\right]$$

implies $\left\| |\psi(T)\rangle - |E_0(1)\rangle \right\| \leq \epsilon$

Satisfiability problems

- Given h: $\{0,1\}^n \rightarrow \{0,1,2,...\}$, is there a value of $z \in \{0,1\}^n$ such that $h(z)=0$?
- Alternatively, what z minimizes $h(z)$?
- Example: 3SAT. $(z_1 \vee z_2 \vee \bar{z}_3) \wedge \cdots \wedge (\bar{z}_{17} \vee z_{37} \vee \bar{z}_{42})$

$$h(z) = \sum_c h_c(z)$$

$$\text{where } h_c(z) = \begin{cases} 0 & \text{clause } c \text{ satisfied by } z \\ 1 & \text{otherwise} \end{cases}$$

Adiabatic optimization

- Define a *problem Hamiltonian* whose ground state encodes the solution:

$$H_P = \sum_{z \in \{0,1\}^n} h(z)|z\rangle\langle z|$$

- Define a beginning Hamiltonian whose ground state is easy to create, for example

$$H_B = -\sum_{j=1}^{n} \sigma_x^{(j)}$$

- Choose $\tilde{H}(s)$ to interpolate from H_B to H_P, for example

$$\tilde{H}(s) = (1-s)H_B + s\,H_P$$

- Choose total run time T so the evolution is nearly adiabatic

[Farhi et al. 2000]

Please mind the gap

Recall rough estimate:

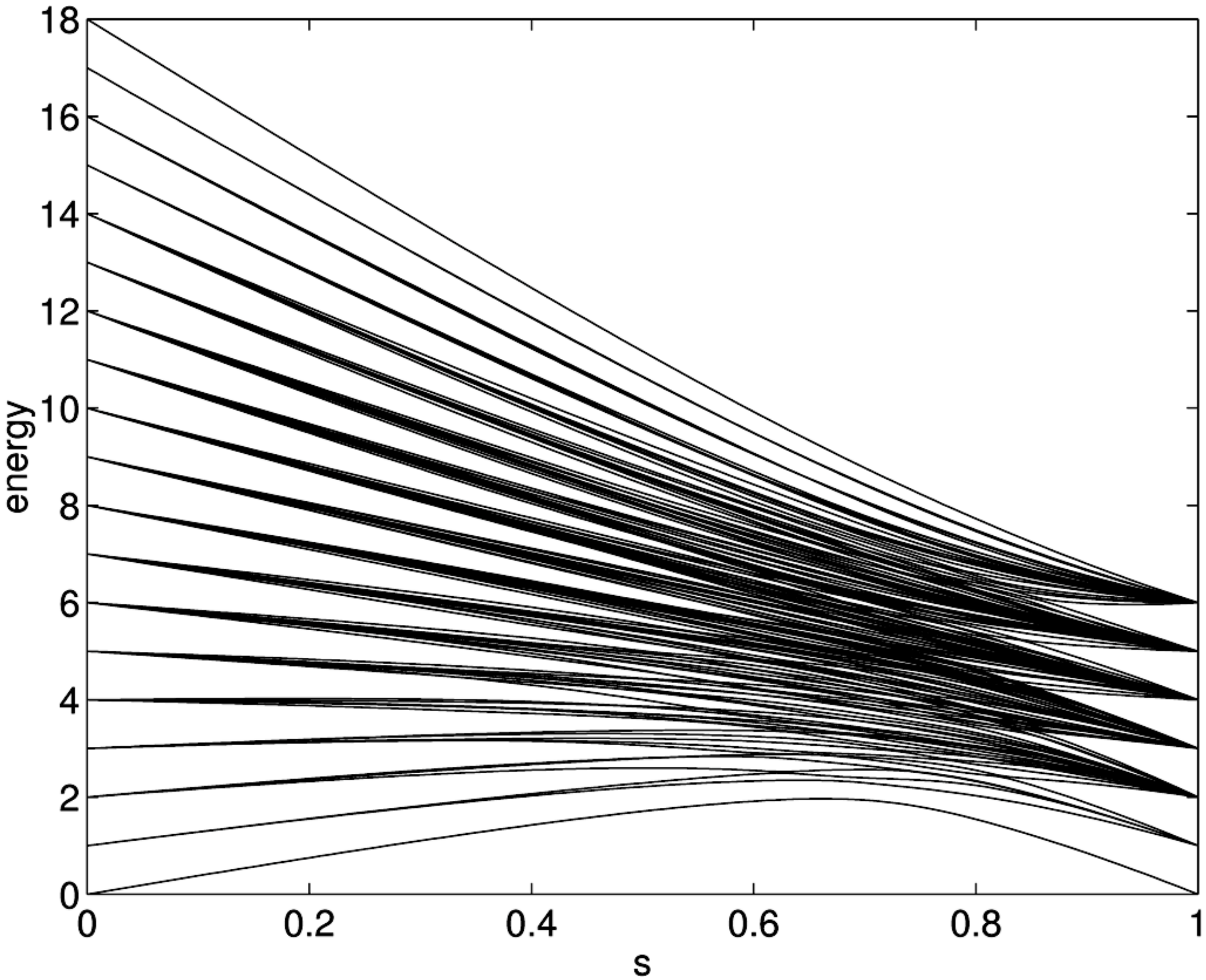
$$T \gg \frac{\Gamma^2}{\Delta^2}\,, \quad \Gamma^2 = \max_{s\in[0,1]} \left\| \left[\dot{\tilde{H}}(s)\right]^2 \right\|$$

For $\tilde{H}(s) = (1-s)H_B + s\,H_P$,

$$\begin{aligned} \|\dot{\tilde{H}}\| &= \|H_P - H_B\| \\ &\leq \|H_B\| + \|H_P\| \end{aligned}$$

Crucial question: How big is Δ?

- ≥1/poly(*n*): Efficient quantum algorithm
- 1/exp(*n*): Inefficient quantum algorithm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Unstructured search

Finding a needle in a haystack: $h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases}$

(here h: {0,1,...,N-1}→{0,1})

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Query complexity (given black box for *h*)

- Classically, $\Theta(N)$ queries
- Quantumly, $O(\sqrt{N})$ queries are sufficient to find *w* [Grover 1996] $(|z\rangle|a\rangle \mapsto |z\rangle|a \oplus h(z)\rangle)$
- This cannot be improved: $\Omega(\sqrt{N})$ queries are necessary [Bennett et al. 1997]

Example: Adiabatic unstructured search

$$h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases} \Rightarrow H_P = \sum_z h(z)|z\rangle\langle z| = 1 - |w\rangle\langle w|$$

Start in $|s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle$

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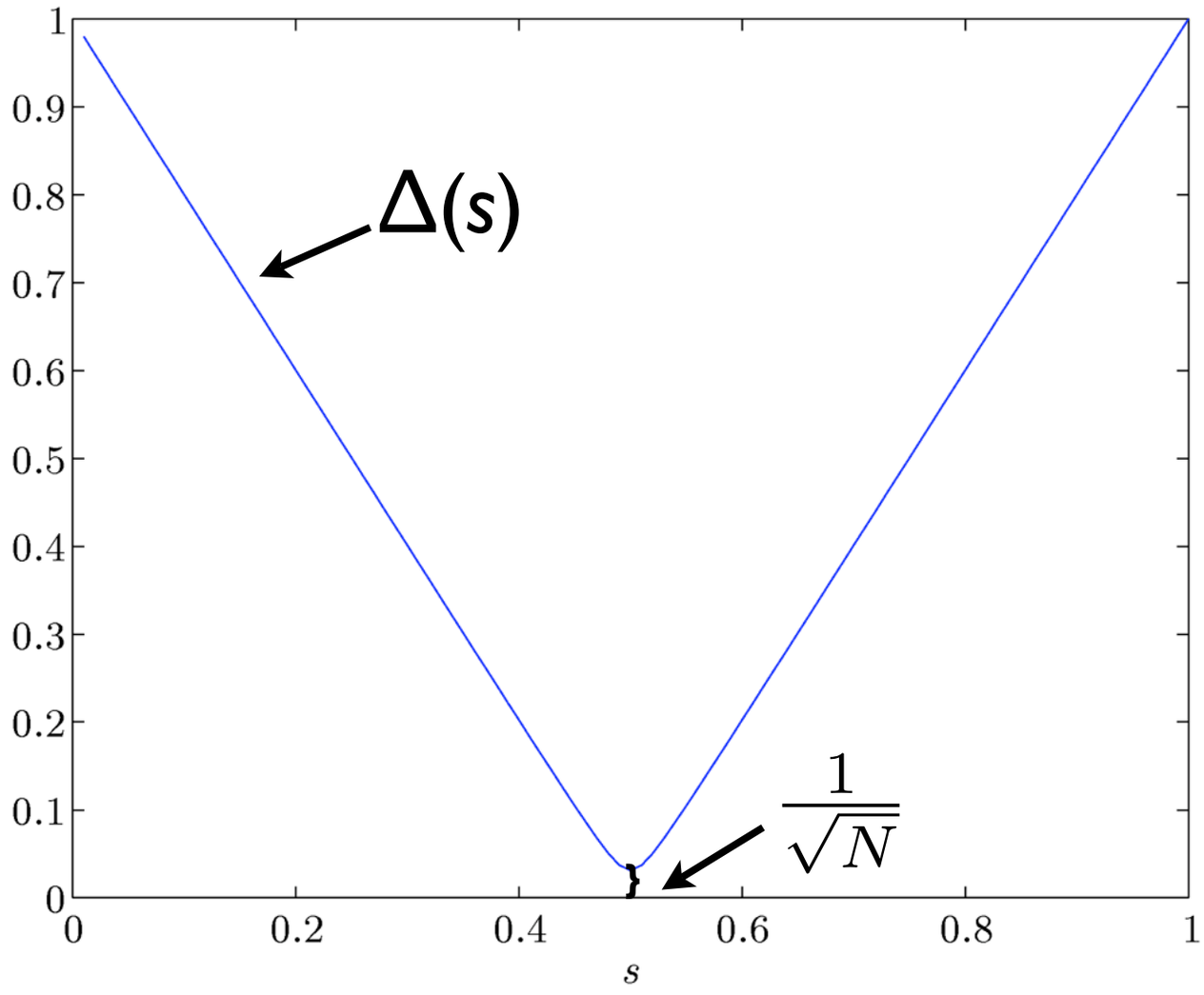

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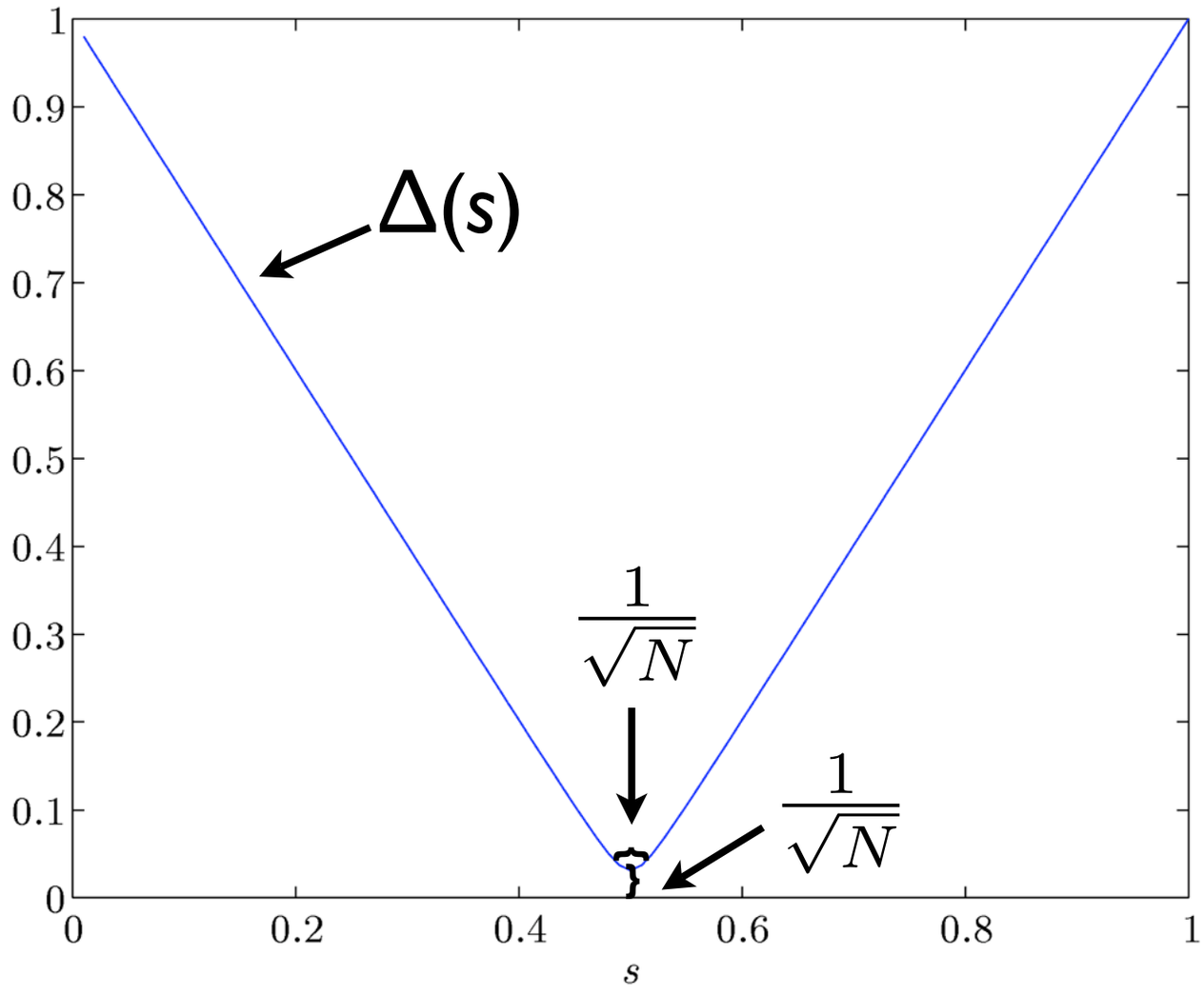

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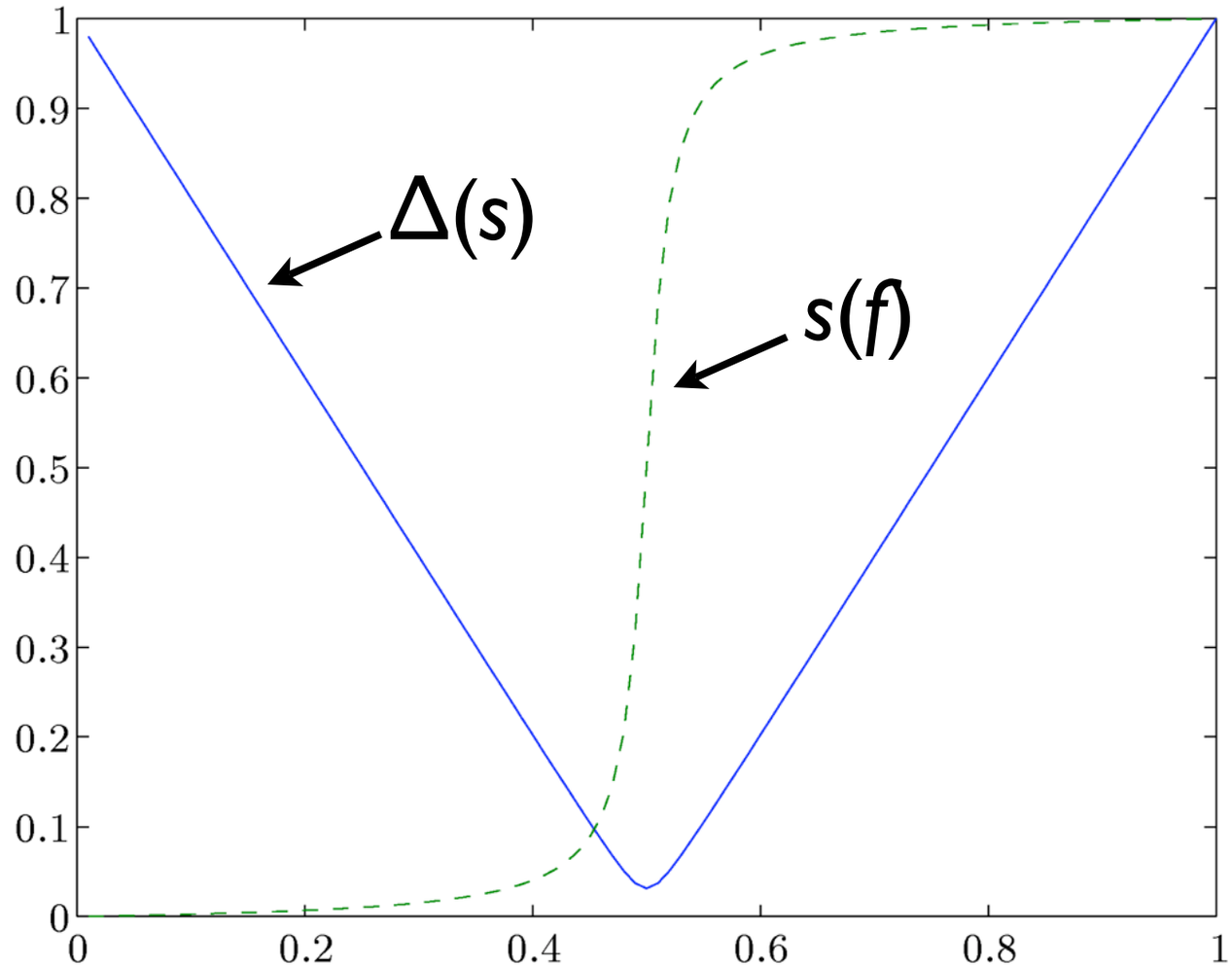
~~$\tilde{H}(s) = (1 - s) H_B + s\, H_P$~~

$$\tilde{H}(s) = [1 - f(s)] H_B + f(s)\, H_P$$



[Roland, Cerf 2002; van Dam et al. 2001]

Example: Transverse Ising model

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{1}{2}\left(1 - \sigma_z^{(j)} \sigma_z^{(j+1)}\right) \qquad \text{“agree”}$$

$$H_B = -\sum_{j=1}^{n} \sigma_x^{(j)} \quad \text{with ground state} \quad |s\rangle = |+\cdots+\rangle$$

$$= \sum_{z \in \{0,1\}^n} |z\rangle$$

$$\tilde{H}(s) = (1-s)H_B + s\, H_P$$

Diagonalize by fermionization (Jordan-Wigner transformation)

Result: $\Delta \propto \frac{1}{n}$ (at critical point of quantum phase transition)

$|E_0(s \approx 0)\rangle \approx |+\cdots+\rangle$

$|E_0(s \approx 1)\rangle \approx \frac{1}{\sqrt{2}}(|0\cdots0\rangle + |1\cdots1\rangle)$

[Farhi et al. 2000]

Example: The Fisher problem

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{J_j}{2}\left(1 - \sigma_z^{(j)}\sigma_z^{(j+1)}\right)$$

J_j=1 or 2, chosen randomly

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[Fisher 1992; Reichardt 2004]

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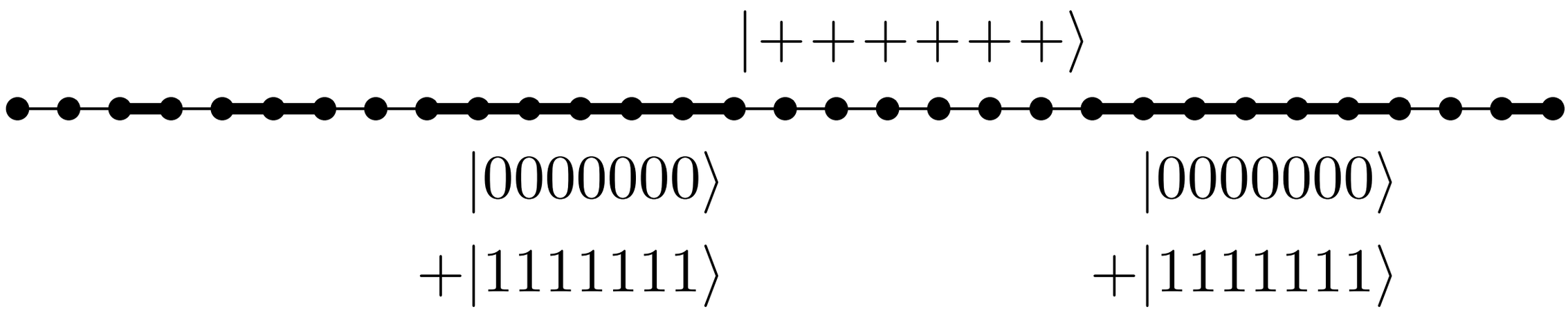


[Fisher 1992; Reichardt 2004]

Robustness of adiabatic QC

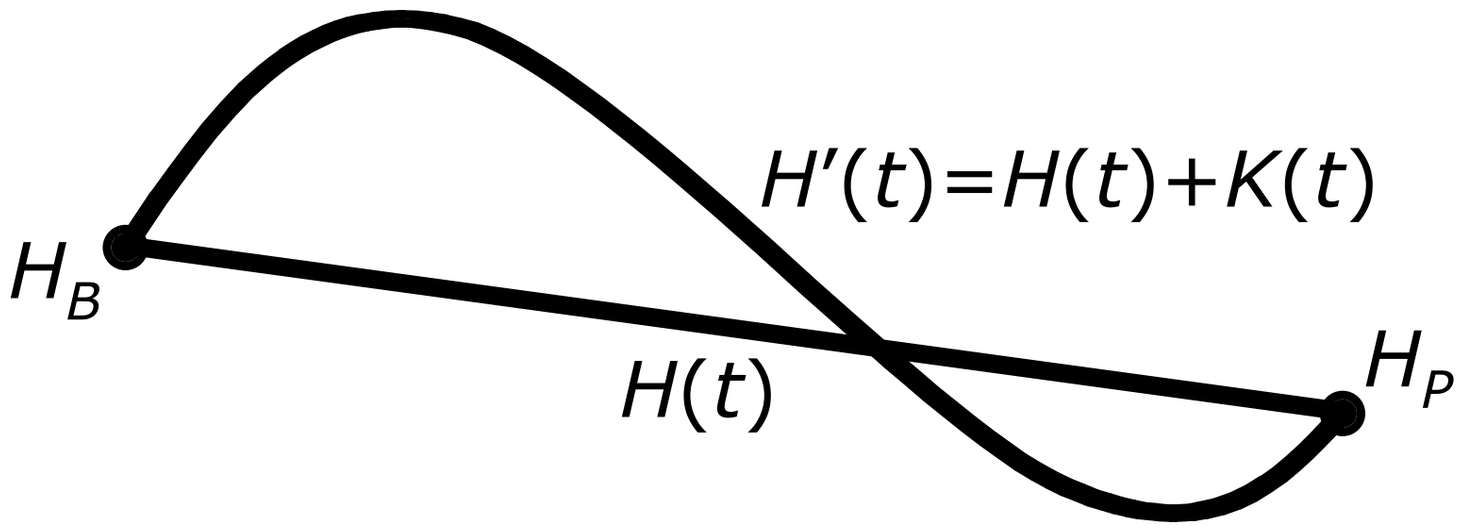
Potential sources of error:

- Unitary control error
- Dephasing in instantaneous eigenstate basis
- Transitions between instantaneous eigenstates: thermal noise

Unitary control error

Adiabatic algorithm depends on going smoothly from H_B to H_P, not on the particular path between them.

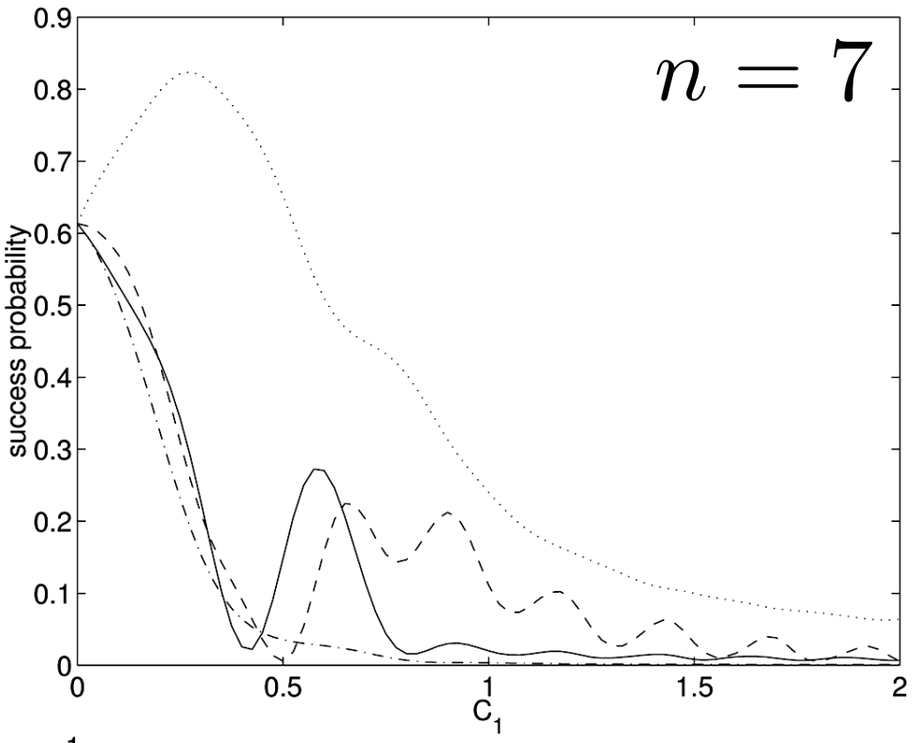
For smooth perturbations, we have no reason to expect the gap will become smaller rather than larger, even if the perturbation is not small (provided it is zero at the beginning and end of the evolution).

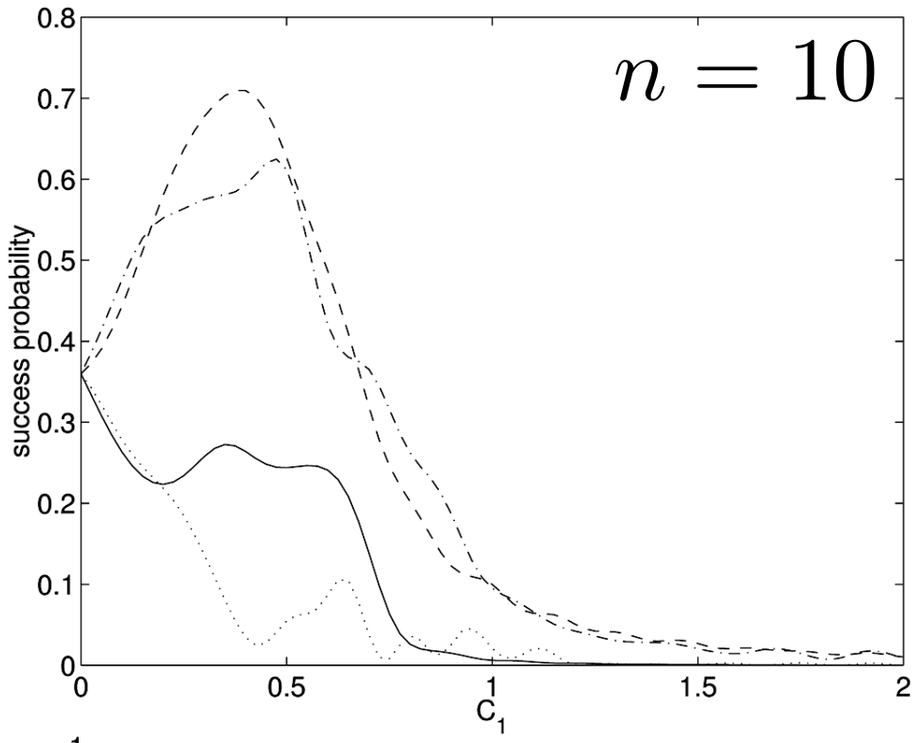


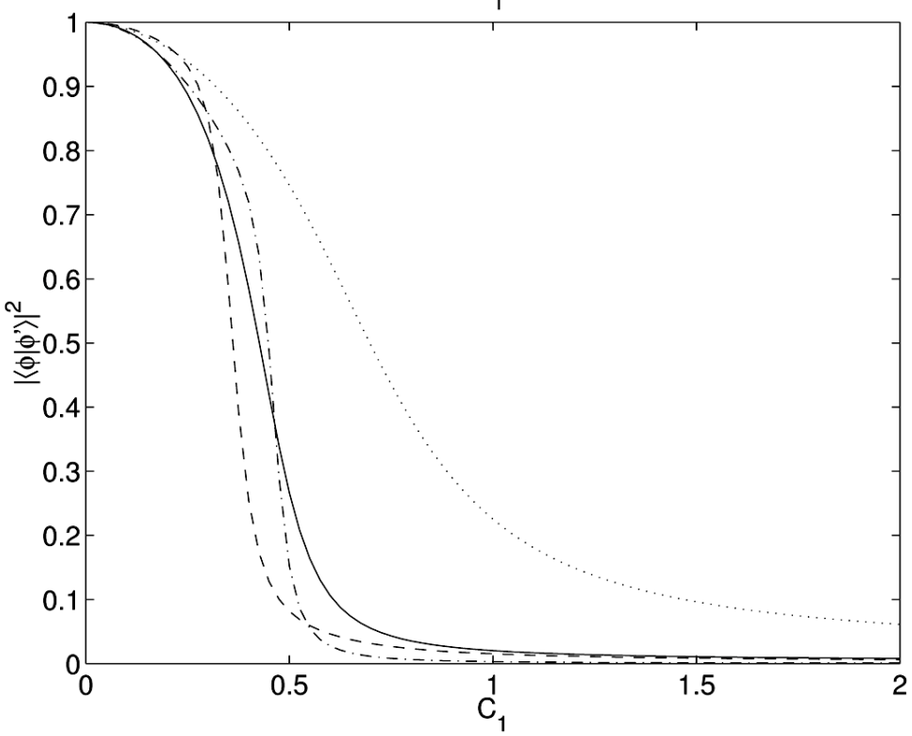
[Childs, Farhi, Preskill 2001]

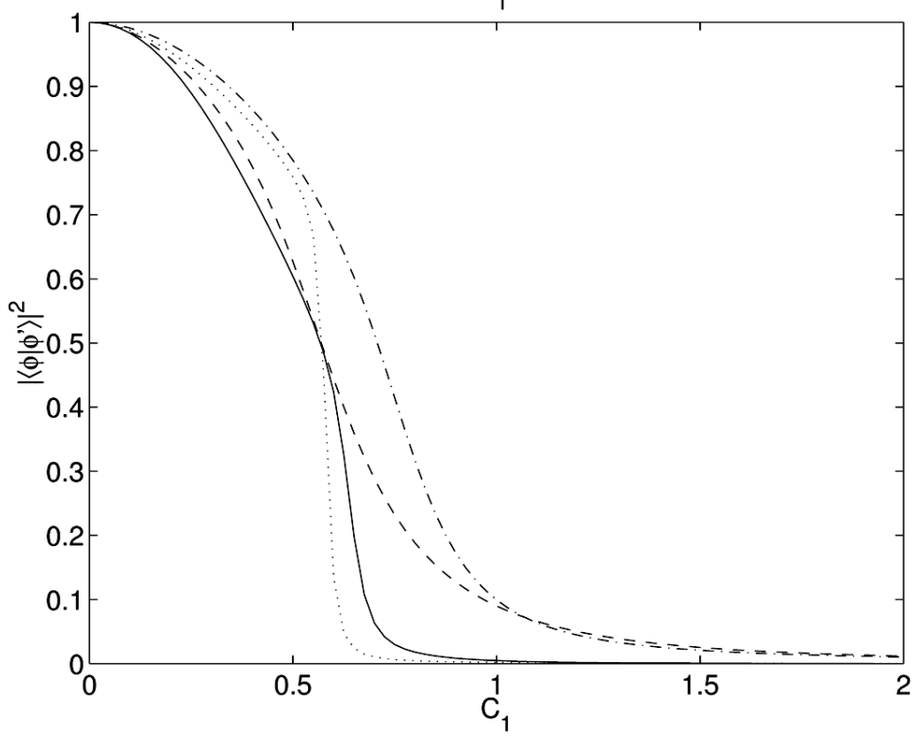
Error in the final Hamiltonian

$$\tilde{K}_1(s) = C_1\, s \sum_{j=1}^{n} \hat{m}_j \cdot \vec{\sigma}^{(j)}$$



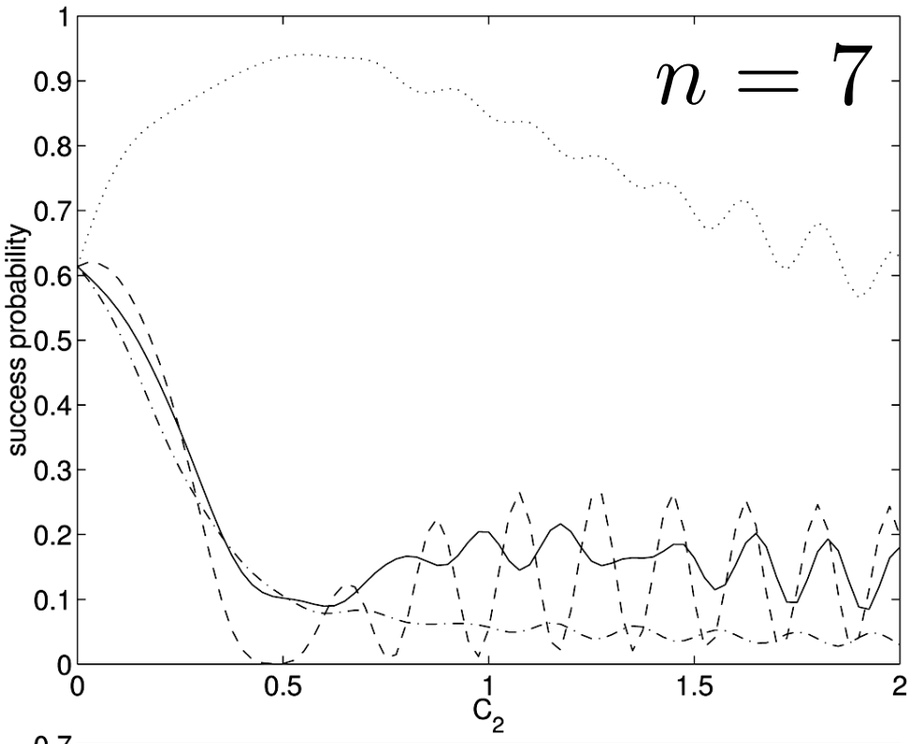


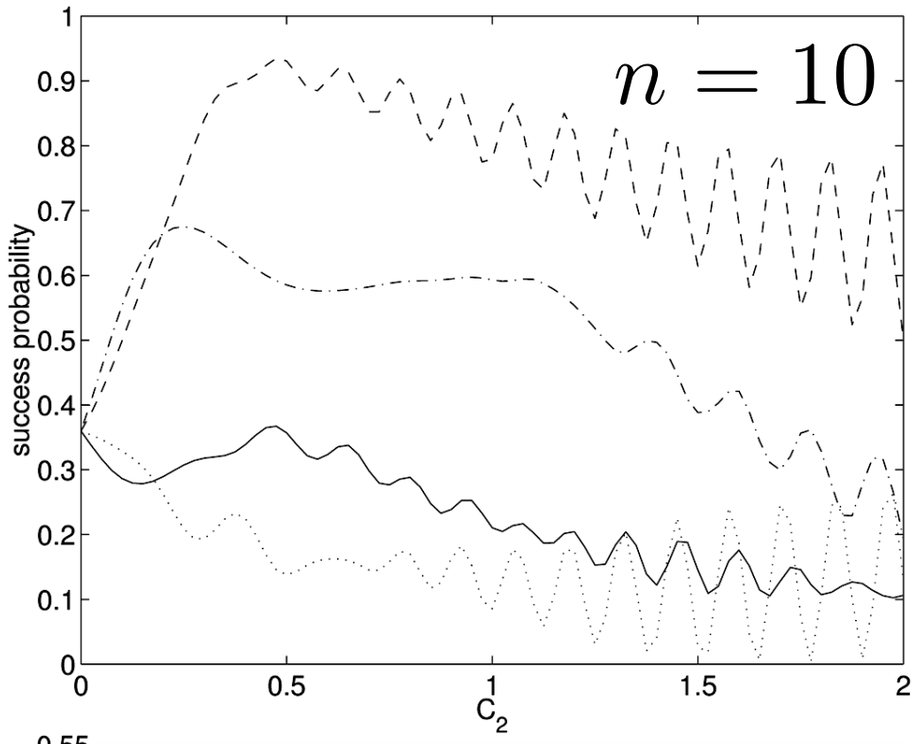


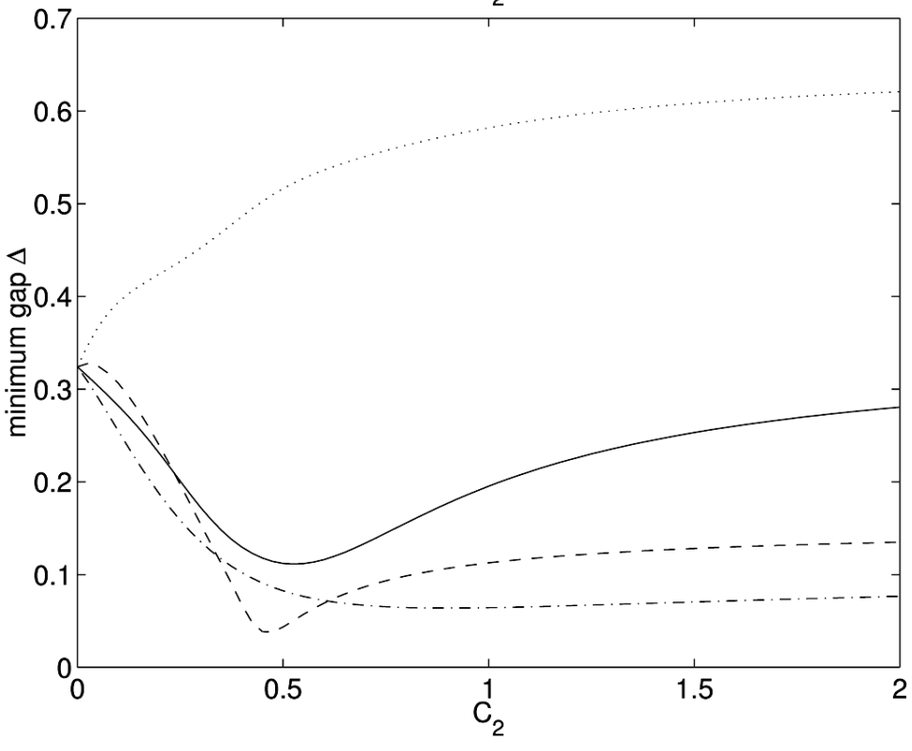


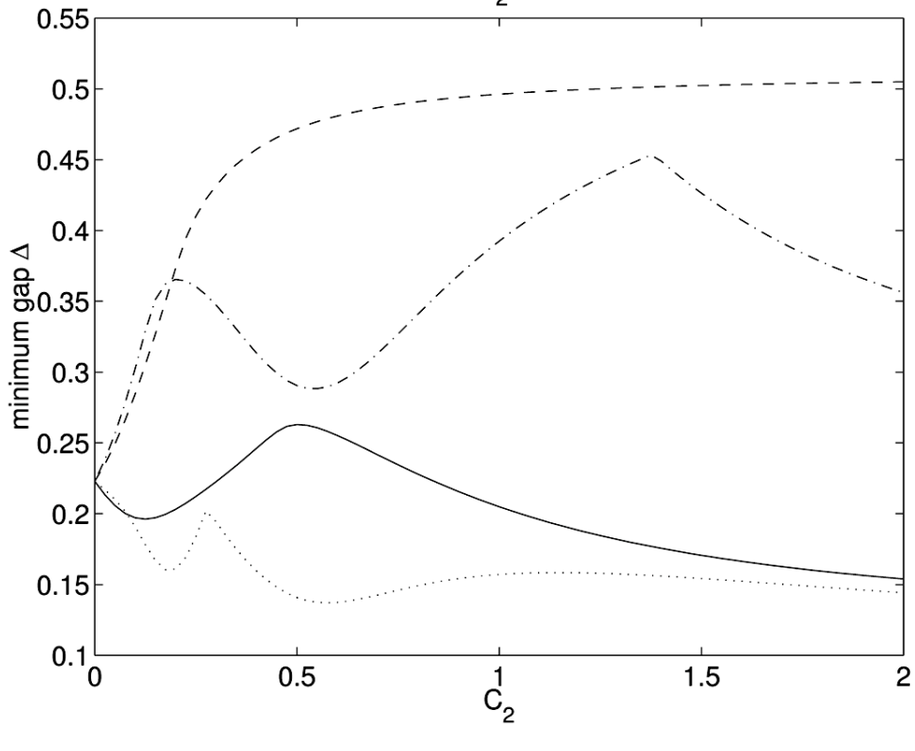
Error in the interpolation

$$\tilde{K}_2(s) = C_2 \sin(\pi s) \sum_{j=1}^{n} \hat{m}_j \cdot \vec{\sigma}^{(j)}$$



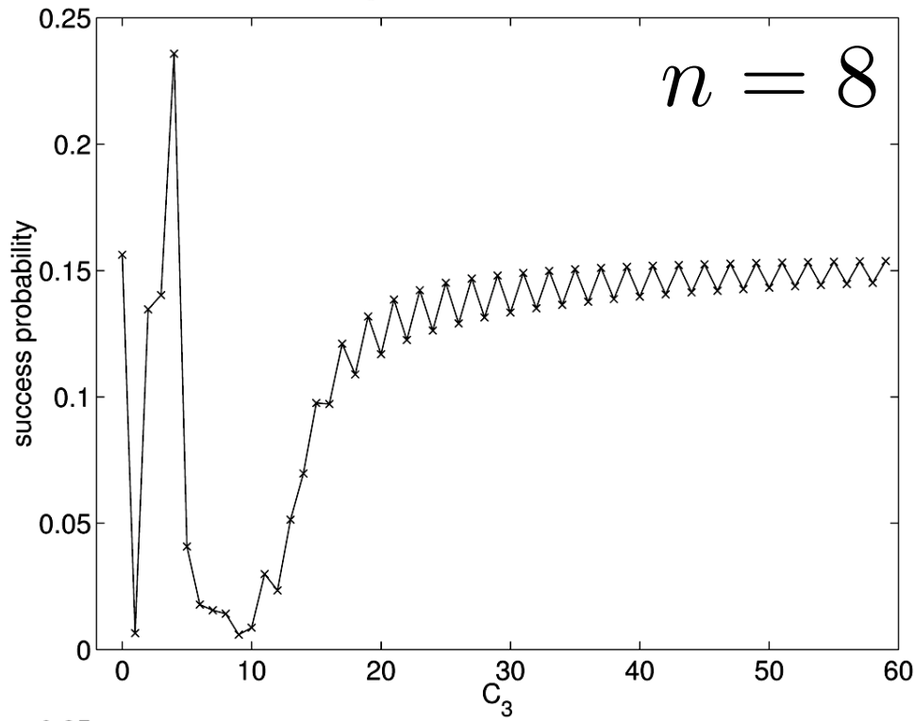


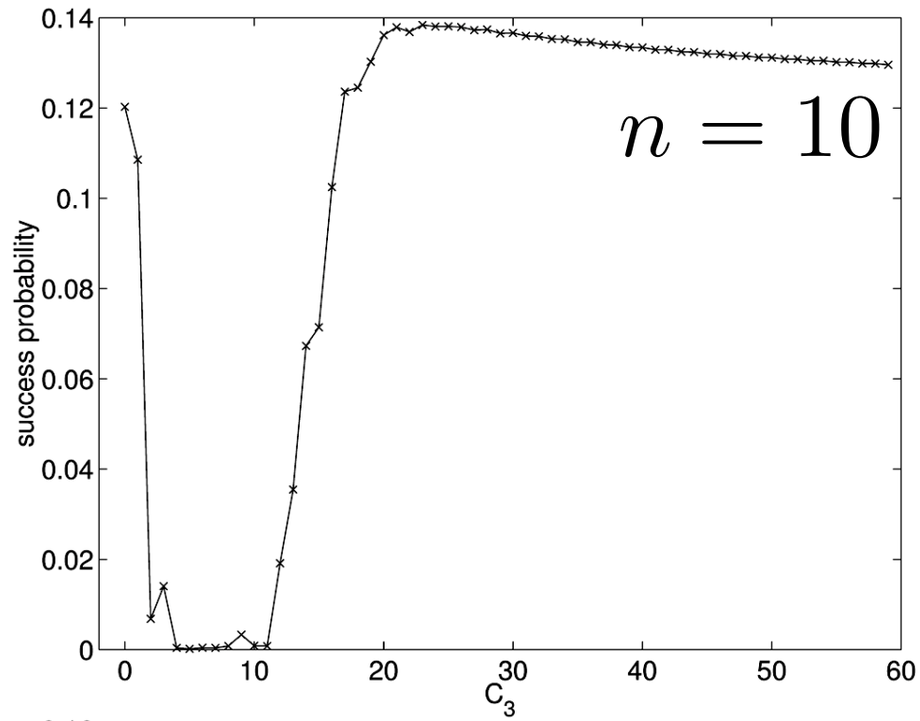


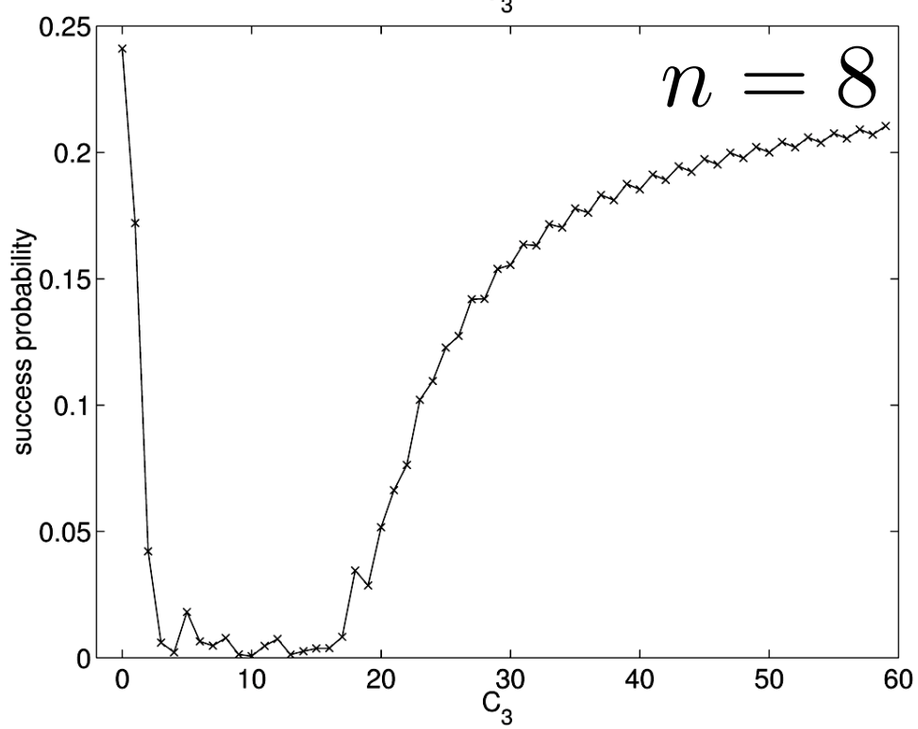


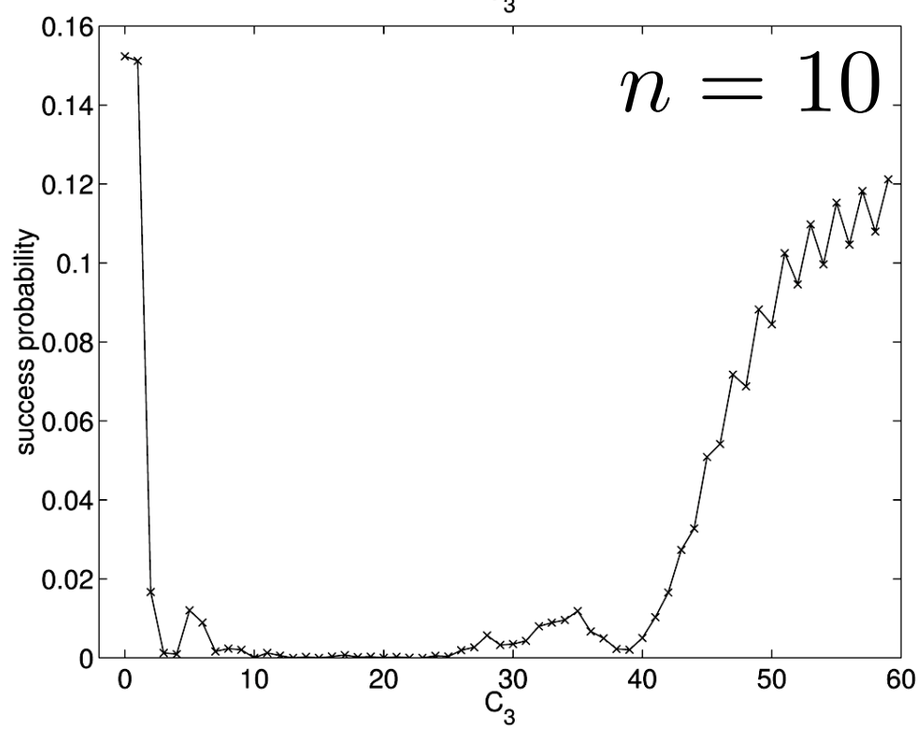
High frequency error

$$\tilde{K}_3(s) = \frac{1}{2}\sin(C_3\pi s)\sum_{j=1}^{n}\hat{m}_j\cdot\vec{\sigma}^{(j)}$$









Thermal noise

Efficient adiabatic quantum computation requires that the minimum gap Δ is not too small.

Provided $k_B T << \Delta$, thermal fluctuations are unlikely to drive the system out of the ground state. So a big gap not only allows for adiabaticity, but also provides protection against thermal noise!

Note: Here it is important that H is the actual Hamiltonian of the of the quantum computer, not just a simulated Hamiltonian.

[Childs, Farhi, Preskill 2001]

Markovian master equations

$$H = H_S + H_E + \lambda V$$

Weak coupling limit: $\lambda \ll 1$

Product initial state: $\rho(0) \otimes \rho_E$

[Davies 1974]: $$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S, \rho] + \lambda^2 K^\natural \rho$$

where

$$K\rho = -\int_0^\infty \mathrm{d}x \ \mathrm{tr}_E[U(-x)VU(x), [V, \rho \otimes \rho_E]]$$

$$K^\natural \rho = \lim_{x\to\infty} \frac{1}{x} \int_0^x \mathrm{d}y \, U(-y)\{K[U(y)\rho U(-y)]\}U(y)$$

$$U(x) = e^{-ix(H_S+H_E)}$$

Markovian master equation for thermal noise

Spins coupled to photons:

$$V = \sum_i \int_0^\infty \mathrm{d}\omega \, [g(\omega) a_\omega \sigma_+^{(i)} + g^*(\omega) a_\omega^\dagger \sigma_-^{(i)}]$$

$$\rho_E = \frac{e^{-\beta H_E}}{\mathrm{tr}\, e^{-\beta H_E}}$$

$$g_{ba} = \begin{cases} \lambda g(\omega_b - \omega_a) & \omega_b > \omega_a \\ 0 & \omega_b \leq \omega_a \end{cases}$$

Then we find

$$N_{ba} = \frac{1}{e^{\beta(\omega_b - \omega_a)} - 1}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S, \rho]$$

$$- \sum_{i,a,b} \big[N_{ba} |g_{ba}|^2 \langle a|\sigma_-^{(i)}|b\rangle\langle b|\sigma_+^{(i)}|a\rangle + (N_{ab}+1)|g_{ab}|^2 \langle b|\sigma_-^{(i)}|a\rangle\langle a|\sigma_+^{(i)}|b\rangle \big]$$

$$\big\{ (|a\rangle\langle a|\rho) + (\rho|a\rangle\langle a|) - 2|b\rangle\langle a|\rho|a\rangle\langle b| \big\}$$

Implications for adiabatic QC

Decoherence terms are suppressed by a factor

$$N_{ba} = \frac{1}{e^{\beta(\omega_b - \omega_a)} - 1} \leq \frac{1}{e^{\beta\Delta} - 1}$$

which is very small provided $\Delta << 1/\beta$.

(Note that this effect is difficult to see in simulations for two reasons:

- Simulating open quantum systems is very computationally intensive, so we can only consider small numbers of qubits.
- Cooling alone may be a good algorithm.)

Question: Is this good enough? I.e., is $T = 1/\text{poly}(n)$ reasonable?

Some questions

- Can we better understand what problems have efficient adiabatic optimization algorithms?
- When can we improve the performance by choosing different interpolation paths?
- Can we increase the robustness of adiabatic quantum computers by careful encoding? In particular, can we make them robust against a small but *n*-independent temperature?