CMSC 330: Organization of Programming Languages

Theory of Regular Expressions

The Theory Behind r.e.'s

- That's it for the basics of Ruby
  - If you need other material for your project, come to office hours or check out the documentation
- Next up: How do r.e.'s really work?
  - Mixture of a very practical tool (string matching with r.e.'s) and some nice theory
  - A great computer science result

A Few Questions about Regular Expressions

- What does a regular expression represent?
  - Just a set of strings
- What are the basic components of r.e.'s?
  - E.g., we saw that e+ is the same as ee*
- How are r.e.'s implemented?
  - We'll see how to build a structure to parse r.e.'s

Definition: Alphabet

- An alphabet is a *finite* set of symbols
  - Usually denoted Σ
- Example alphabets:
  - Binary: $\Sigma = \{0,1\}$
  - Decimal: $\Sigma = \{0,1,2,3,4,5,6,7,8,9\}$
  - Alphanumeric: $\Sigma = \{0\text{-}9, a\text{-}z, A\text{-}Z\}$

Definition: String

- A *string* is a finite sequence of symbols from Σ
  - ε is the empty string ("" in Ruby)
  - $|s|$ is the length of string s
    - $|Hello| = 5$, $|\varepsilon| = 0$
  - Note: $\varnothing$ is the empty set (with 0 elements); $\varnothing \neq \{ \varepsilon \}$

- Example strings:
  - $0101 \in \Sigma = \{0,1\}$ (binary)
  - $0101 \in \Sigma =$ decimal
  - $0101 \in \Sigma =$ alphanumeric

Definition: Concatenation

- *Concatenation* is indicated by juxtaposition
  - If s_1 = super and s_2 = hero, then s_1s_2 = superhero
  - Sometimes also written $s_1 \cdot s_2$
  - For any string s, we have $s\varepsilon = \varepsilon s = s$
  - You *can* concatenate strings from different alphabets, then the new alphabet is the union of the originals:
    - If s_1 = super $\in \Sigma_1 = \{s,u,p,e,r\}$ and s_2 = hero $\in \Sigma_2 = \{h,e,r,o\}$, then s_1s_2 = superhero $\in \Sigma_3 = \{e,h,o,p,r,s,u\}$

Definition: Language

- A *language* is a set of strings over an alphabet
- Example: The set of phone numbers over the alphabet Σ = {0, 1, 2, 3, 4, 5, 6, 7, 9, (,), -}
  - Give an example element of this language `(123) 456-7890`
  - Are all strings over the alphabet in the language? `No`
  - Is there a Ruby regular expression for this language?

    `/\(\d{3,3}\) \d{3,3}-\d{4,4}/`
- Example: The set of all strings over Σ
  - Often written Σ*

Languages (cont'd)

- Example: The set of strings of length 0 over the alphabet Σ = {a, b, c}
  - $\{s \mid s \in \Sigma^* \text{ and } |s| = 0\} = \{\varepsilon\} \neq \varnothing$
- Example: The set of all valid Ruby programs
  - Is there a Ruby regular expression for this language?

  No. Matching brackets so they are balanced is impossible.
  { { { } } } or $\{^3\ \}^3$ or, in general, $\{^n\ \}^n$
- Can r.e.'s represent all possible languages?
  - The answer turns out to be no!
  - The languages represented by regular expressions are called, appropriately, the regular languages

Operations on Languages

- Let Σ be an alphabet and let L, L_1, L_2 be languages over Σ
- Concatenation L_1L_2 is defined as
  - $L_1L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$
  - Example: L_1 = {"hi", "bye"}, L_2 = {"1", "2"}
    - L_1L_2 = {"hi1", "hi2", "bye1", "bye2"}
- Union is defined as
  - $L_1 \cup L_2 = \{ x \mid x \in L_1 \text{ or } x \in L_2\}$
  - Example: L_1 = {"hi", "bye"}, L_2 = {"1", "2"}
    - $L_1 \cup L_2$ = {"hi", "bye", "1", "2"}

Operations on Languages (cont'd)

- Define L^n inductively as
  - $L^0 = \{\varepsilon\}$
  - $L^n = LL^{n-1}$ for $n > 0$
- In other words,
  - $L^1 = LL^0 = L\{\varepsilon\} = L$
  - $L^2 = LL^1 = LL$
  - $L^3 = LL^2 = LLL$
  - ...

Examples of L^n

- Let $L = \{a, b, c\}$
- Then
  - $L^0 = \{\varepsilon\}$
  - $L^1 = \{a, b, c\}$
  - $L^2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

Operations on Languages (cont'd)

- *Kleene closure* is defined as

  $L^* = \bigcup_{i \in [0..\infty]} L^i$

- In other words...

  L^* is the language (set of all strings) formed by concatenating together zero or more strings from L

Definition of Regexps

- Given an alphabet Σ, the *regular expressions* over Σ are defined inductively as

regular expression	denotes language
$\varnothing$	$\varnothing$
ε	$\{\varepsilon\}$
each element $\sigma \in \Sigma$	$\{\sigma\}$

  – ...

Definition of Regexps (cont'd)

- Let A and B be regular expressions denoting languages L_A and L_B, respectively

regular expression	denotes language
AB	$L_A L_B$
$(A\|B)$	$L_A \cup L_B$
A^*	L_A^*

- There are no other regular expressions over Σ
- We use ()'s as needed for grouping

The Language Denoted by an r.e.

- For a regular expression e, we will write [[e]] to mean the language denoted by e
  - [[a]] = {a}
  - [[(a|b)]] = {a, b}

- If s∈[[re]], we say that re *accepts*, *describes*, or *recognizes* s.

Example 1

- All strings over Σ = {a, b, c} such that all the a's are first, the b's are next, and the c's last
  - Example: aaabbbbccc but not abcb
- Regexp: a*b*c*
  - This is a valid regexp because:
    - a is a regexp ([[a]] = {a})
    - a* is a regexp ([[a*]] = {ε, a, aa, ...})
    - Similarly for b* and c*
    - So a*b*c* is a regular expression

    (Remember that we need to check this way because regular expressions are defined inductively.)

Which Strings Does a*b*c* Recognize?

aabbbcc

Yes; aa ∊ [[a*]], bbb ∊ [[b*]], and cc ∊ [[c*]], so entire string is in [[a*b*c*]]

abb

Yes, abb = abbε, and ε ∊ [[c*]]

ac

Yes

ε

Yes

aacbc

No

abcd

No -- outside the language

Example 2

- All strings over Σ = {a, b, c}
- Regexp: (a|b|c)*
- Other regular expressions for the same language?
  - (c|b|a)*
  - (a*|b*|c*)*
  - (a*b*c*)*
  - ((a|b|c)*|abc)
  - etc.

Example 3

- All whole numbers containing the substring 330
- Regular expression: (0|1|...|9)*330(0|1|...|9)*
- What if we want to get rid of leading 0's?
- ((1|...|9)(0|1|...|9)*330(0|1|...|9)* | 330(0|1|...|9)*)
- Any other solutions?

- Challenge: What about all whole numbers not containing the substring 330?
  – Is it recognized by a regexp?

Yes. We'll see how to find it later…

Example 4

- What is the English description for the language that (10|0)*(10|1)* denotes?
  – (10|0)*
    - 0 may appear anywhere
    - 1 must always be followed by 0
  – (10|1)*
    - 1 may appear anywhere
    - 0 must always be preceded by 1
  – Put together, all strings of 0's and 1's where every pair of adjacent 0's precedes any pair of adjacent 1's

What Strings are in (10|0)*(10|1)* ?

00101000 110111101

First part in [[(10|0)*]]

Second part in [[(10|1)*]]

Notice that 0010 also in [[(10|0)*]]

But remainder of string is not in [[(10|1)*]]

0010101

Yes

101

Yes

011001

No

Example 5

- What language does this regular expression recognize?
  - ((1|ε)(0|1|...|9) | (2(0|1|2|3))) : (0|1|...|5)(0|1|...|9)

- All valid times written in 24-hour format
  - 10:17
  - 23:59
  - 0:45
  - 8:30

Two More Examples

- (000|00|1)*
  - Any string of 0's and 1's with no single 0's
- (00|0000)*
  - Strings with an even number of 0's
  - Notice that some strings can be accepted more than one way
    - 000000 = 00·00·00 = 00·0000 = 0000·00
  - How else could we express this language?
    - (00)*
    - (00|000000)*
    - (00|0000|000000)*
    - etc…

Regular Languages

- The languages that can be described using regular expressions are the *regular languages* or *regular sets*
- Not all languages are regular
  - Examples (without proof):
    - The set of palindromes over Σ
    - $\{a^n b^n \mid n > 0\}$ (a^n = sequence of n a's)
- Almost all programming languages are not regular
  - But aspects of them sometimes are (e.g., identifiers)
  - Regular expressions are commonly used in parsing tools

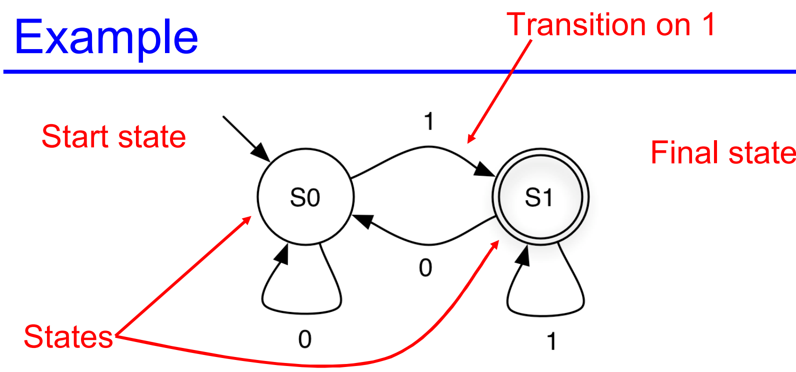
Ruby Regular Expressions

- Almost all of the features we've seen for Ruby r.e.'s can be reduced to this formal definition
  - /Ruby/ – concatenation of single-character r.e.'s
  - /(Ruby|Regular)/ – union
  - /(Ruby)*/ – Kleene closure
  - /(Ruby)+/ – same as (Ruby)(Ruby)*
  - /(Ruby)?/ – same as (ε|(Ruby)) (// is ε)
  - /[a-z]/ – same as (a|b|c|...|z)
  - / [^0-9]/ – same as (a|b|c|...) for a,b,c,... ∈ Σ - {0..9}
  - ^, $ – correspond to extra characters in alphabet

Implementing Regular Expressions

- We can implement regular expressions by turning them into a *finite automaton*
  - A "machine" for recognizing a regular language

Example



- Machine starts in *start* or *initial* state
- Repeat until the end of the string is reached:
  - Scan the next symbol s of the string
  - Take transition edge labeled with s
- The string is *accepted* if the automaton is in a *final* or *accepting* state when the end of the string is reached

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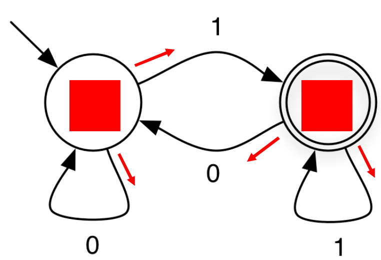
Example



0 0 1 0 1 1

accepted

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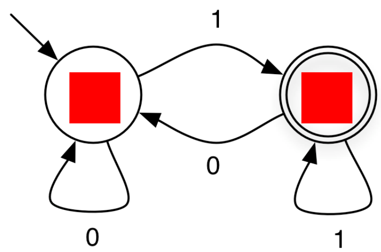
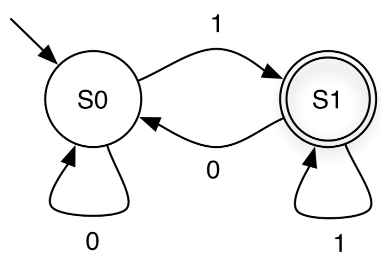
Example



0 0 1 0 1 0

not accepted

What Language is This?



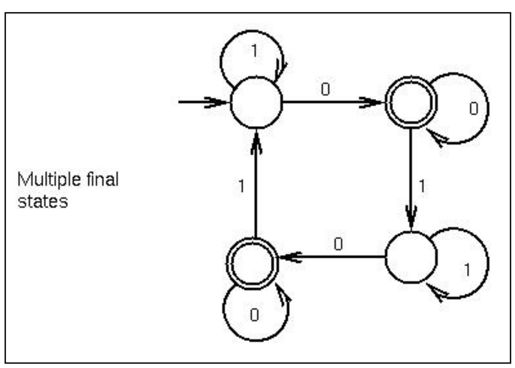
- All strings over {0, 1} that end in 1
- What is a regular expression for this language?
  - (0|1)*1

Formal Definition

- A *deterministic finite automaton (DFA)* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
    - the strings recognized by the DFA are over this set
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
    - How many can there be?
  - $\delta : Q \times \Sigma \rightarrow Q$ specifies the DFA's transitions
    - What's this definition saying that δ is?

More on DFAs

- A finite state automata can have more than one final state:

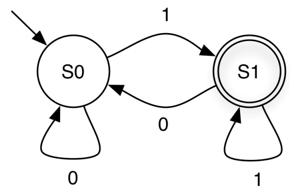


- A string is accepted as long as there is at least one path to a final state

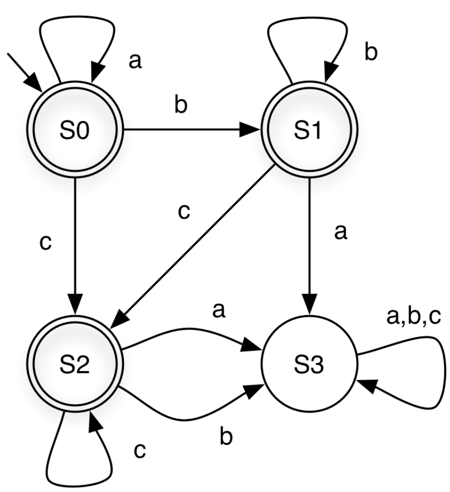
Our Example, Formally

- Σ = {0, 1}
- Q = {S0, S1}
- q_0 = S0
- F = {S1}
-

δ	0	1
S0	S0	S1
S1	S0	S1



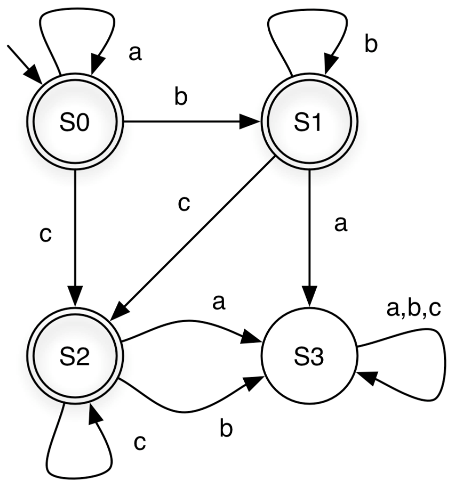
Another Example



string	state at end	accepts ?
aabcc	S2	Y
acc	S2	Y
bbc	S2	Y
aabbb	S1	Y
aa	S0	Y
ε	S0	Y
acba	S3	N

(a,b,c notation shorthand for three self loops)

Another Example (cont'd)

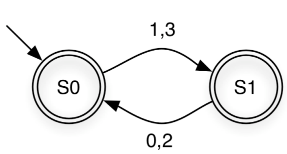


What language does this DFA accept? a*b*c*

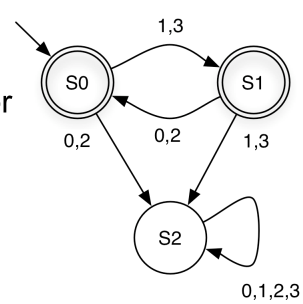
S3 is a *dead state* – a nonfinal state with no transition to another state

Shorthand Notation

- If a transition is omitted, assume it goes to a dead state that is not shown



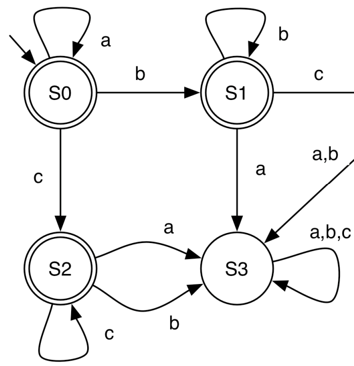
is short for



Language?

Strings over {0,1,2,3} with alternating even and odd digits, beginning with odd digit

What Lang. Does This DFA Accept?



a*b*c* again, so DFAs are not unique

Practice

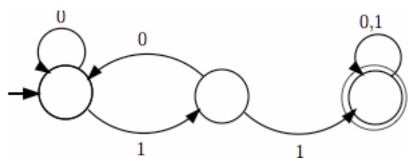
Give the English descriptions and the DFA or regular expression of the following languages:

- ((0|1)(0|1)(0|1)(0|1)(0|1))*

  `all strings with length a multiple of 5`

- (01)*|(10)*|(01)*0|(10)*1

  `all alternating binary strings`



`all binary strings containing the substring "11"`

Practice

Give the regular expressions and DFAs for the following languages:

- You and your neighbors' names
- All valid DNA strings (including only ACGT and appearing in multiples of 3)
- All binary strings containing an even length substring of all 1's
- All binary strings containing exactly two 1's
- All binary strings that start and end with the same number