CO639 Scribe Notes

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CNOT

$$
\begin{aligned}
X \otimes I &\to X \otimes X \\
Z \otimes I &\to Z \otimes Z \\
I \otimes X &\to I \otimes X \\
I \otimes Z &\to Z \otimes Z
\end{aligned}
$$

Stabilizer for 2 qubits encoded with 7 qubit code

$$
\begin{array}{c|c}
XXXXIII & \\
XXIIXXI & \\
XIXIXIX & I \\
ZZZZIII & \\
ZZIIZZI & \\
ZIZIZIZ & \\
\hline
 & XXXXIII \\
 & XXIIXXI \\
I & XIXIXIX \\
 & ZZZZIII \\
 & ZZIIZZI \\
 & ZIZIZIZ
\end{array}
$$

Applying *CNOT* transversally to qubits encoded with 7 qubit code

$$
\begin{array}{c|c}
XXXXIII & XXXXIII \\
XXIIXXI & XXIIXXI \\
XIXIXIX & XIXIXIX \\
ZZZZIII & IIIIIII \\
ZZIIZZI & IIIIIII \\
ZIZIZIZ & IIIIIII \\
\hline
IIIIIII & XXXXIII \\
IIIIIII & XXIIXXI \\
IIIIIII & XIXIXIX \\
ZZZZIII & ZZZZIII \\
ZZIIZZI & ZZIIZZI \\
ZIZIZIZ & ZIZIZIZ
\end{array}
$$

Notice that all rows are in the original stabilizer (ie. either generators or products of generators).

$CNOT^{\otimes 7}$

$$\begin{aligned} \overline{X} \otimes \overline{I} &\rightarrow \overline{X} \otimes \overline{X} \\ \overline{Z} \otimes \overline{I} &\rightarrow \overline{Z} \otimes \overline{Z} \\ \overline{I} \otimes \overline{X} &\rightarrow \overline{I} \otimes \overline{X} \\ \overline{I} \otimes \overline{Z} &\rightarrow \overline{Z} \otimes \overline{Z} \end{aligned}$$

Thus, $CNOT^{\otimes 7} = \overline{CNOT}$ (logical)

CSS Codes

- For CSS Codes, $CNOT^{\otimes n}$ is an encoded $\overline{CNOT}^{\otimes k}$
- Conversely, if $CNOT^{\otimes n}$ is an encoded $\overline{CNOT}^{\otimes k}$ then we have a CSS Code

For the 5-qubit code:

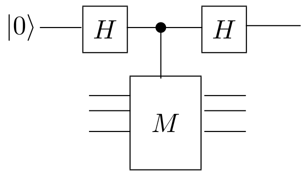
$$\begin{array}{c|c} XZZXI & YXXYI \\ IXZZX & IYXXY \\ XIXZZ & YIYXX \\ ZXIXZ & XYIYX \\ ZZXIX & XXYIY \end{array}$$

$$T : X \rightarrow Y, Z \rightarrow X \Longrightarrow T^{\otimes 5} = \overline{T}$$

$GF(4)$
$T : 1 \rightarrow \omega^2 \rightarrow \omega$
Multiplication by $\omega^2 \leftrightarrow T$
Linear $GF(4) \leftrightarrow T^{\otimes n}$ encoded operation

Knill's Theorem: $\mathcal{C}$ can be simulated classically (if starting with $|00..0\rangle$)
proof: initial state has stabilizer $Z_1, Z_2, ...Z_n$. Each gate from $\mathcal{C}$ transforms stabilizer, but still needs only $O(n^2)$ bits to record.

Measurement
Non fault-tolerant syndrome measurement uses eigenvalue kickback.



The ancilla qubit, at the end, is equal to the eigenvalue of M. This is a syndrome bit if $M \in \mathcal{S}$.

However, the ancilla qubit interacts with multiple data qubits. Thus, an error during syndrome measurement can cause multiple errors in the data block. This is not fault tolerant.

A possible solution is to use a cat state:

$$|00...0\rangle + |11...1\rangle$$

This is the Shor method of error correction. The cat state must be large enough that each qubit from the cat state interacts with at most one qubit in the data block. Then, this is a fault tolerant measurement.

Performing a transversal Hadamard on the cat state produces a superposition of only even weighted strings,

$$H^n(|00...0\rangle + |11...1\rangle) = \sum |even\rangle$$

If the relative phase in the cat state were negative, this would produce a superposition of only odd weighted strings,

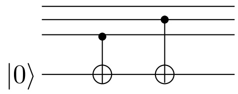
$$H^n(|00...0\rangle - |11...1\rangle) = \sum |odd\rangle$$

Thus, after the transversal interaction with the data block, we acquire the syndrome bit by performing a transversal Hadamard, measuring, then checking the parity of the measured string.

Problems with Cat State:

1. How to build the cat state in a fault tolerant way?
2. A single *phase* error destroys the measured bit

We can address the first problem by verifying the cat state prior to its use.



Thus, after we build the cat state with any procedure, we simply compare pairs of qubits to ensure that the state is what we intended.

For the second problem, we can use repeated measurements.