Quantum Error Correction

Notes for lecture 11

Prepared by Casey Myers
Edited by Daniel Gottesman

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Steane Method (for CSS codes)

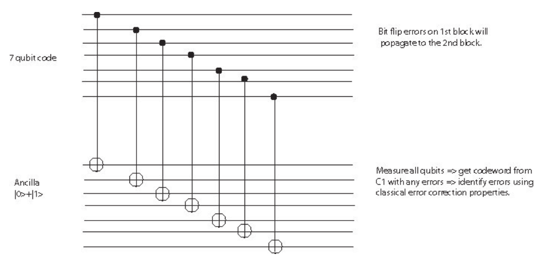

Figure 1: Note: the cat state $|0\rangle + |1\rangle \equiv |\overline{0}\rangle + |\overline{1}\rangle$.

$$|\overline{0}\rangle + |\overline{1}\rangle = \sum_{u \in C_1/C_2^{\perp}} \sum_{v \in C_2^{\perp}} |u+v\rangle = \sum_{u \in C_1} |u\rangle$$

$u \in C_1$, $C_2^{\perp} \subseteq C_1$.
E.g. 1 encoded block: measure all qubits. $|\overline{u}\rangle \rightarrow \sum_{w \in C_2^{\perp}} |u+w\rangle$. Get $u+w$ for random $w \in C_2^{\perp}$. Logical codewords $\Leftrightarrow u \in C_1/C_2^{\perp}$
$\Rightarrow$ Measurement identifies bit flip error and encoded state (in Z basis).

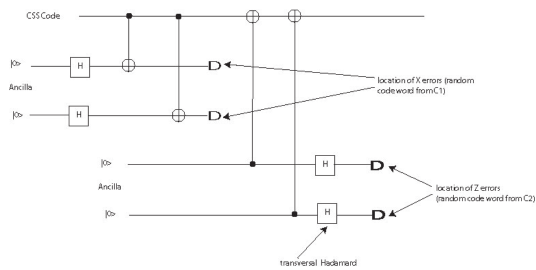

Figure 2: Note: $|0\rangle \equiv |\bar{0}\rangle$.

In the figures, each line represents n qubits encoded via CSS code. If there are no errors, we just get a random codeword from C_1. If there are bit flip errors in the data, they propagate forward along the CNOTs to the ancilla, and after measurement we get a random codeword of C_1 with an error in the appropriate location. Phase errors propagate backwards along the CNOTs in the second figure, and performing the Hadamard gives us a superposition of codewords from C_2, with bit flip errors in the locations of the phase errors from the data block.
- Repeat measurement
- Verify ancillas (to make sure we don't have multiple bit flip errors). In the ancilla for phase errors, bit flip errors will propagate into the data. In the ancilla for bit flip errors, the Hadamard transform turns initial bit flip errors into phase errors, which can also propagate into the data.

Ancilla Purification

Tells us:

1.) Individual bit flip errors.

2.) Encoded bit flip errors.

Either discard bad ancillas or correct.

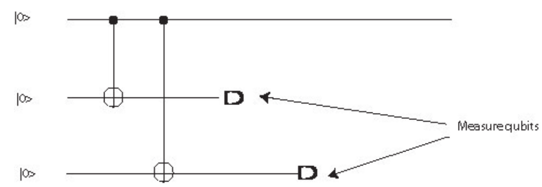

Figure 3: Note: $|0\rangle \equiv |\overline{0}\rangle$.

Measurements

How does stabilizer, $\overline{X}, \overline{Z}$ change under measurement?
Measure $N \in \mathcal{P}$

1.) $N \in S \ \Rightarrow$ nothing happens.

2.) $N \in N(S) \setminus S \ \Rightarrow$ measures logical Pauli in $N(S) \setminus S$.

3.) $N \notin N(S) \ \Rightarrow \ \exists M_1 \in S,\ \{M_1, S\} = 0$.

New stabilizer S':
$\pm N \in S'$. ($+N$ if measure $+1$, $-N$ if measure -1)
$M_1 \notin S'$
Generators $M_2, \ldots, M_r \in S$

$$\begin{aligned} [M_i, N] &= 0 \Rightarrow M_i \in S' \\ \{M_i, N\} &= 0 \Rightarrow M_1 M_i \text{ commutes with } N \Rightarrow M_1 M_i \in S' \end{aligned}$$

$\overline{X}$, $\overline{Z}$ etc.: $M_1\overline{X} = \overline{X} \ \Rightarrow$ choose coset representatives that commute with N.
r generators of S', $\pm N, (M_i$ OR $M_1 M_i)$.
$n - r$ $\overline{X}$'s, $n - r$ $\overline{Z}$'s.

Map $-N$ to $+N$:

Perform M_1 on state, commutes with M_1 and $\overline{X}, \overline{Z}$, but anticommutes with N, so changes the sign of N without altering anything else in the new stabilizer.

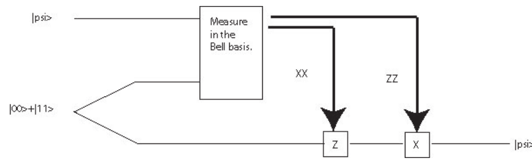

Figure 4: Example: Teleportation

	IXX	Measure XXI	IXX	Measure ZZI	$\pm ZZI$		
	IZZ		$\pm XXI$		XXI		
$\overline{X}$:	XII	$\longrightarrow$	XII	$\longrightarrow$	XXX	$\equiv$	IIX
$\overline{Z}$:	ZII	Here $M_1 = IZZ$	ZZZ	Here $M_1 = IXX$	ZZZ	$\equiv$	IIZ

Table 1: Analysis of teleportation

Look at M_1: what do we have to do to $M_1 N$ (E.g. $\pm XXI$) to take it to + (i.e. $-N \to +N$)? Do IZZ but only need to look at the 3rd qubit: Z. Similarly for $\pm ZZI$: IXX OR just X.

Suppose we can do a CNOT gate but want a P gate:
CNOT $\to$ P gate and Pauli measurements. ($P:\ X \to Y, Z \to Z$)

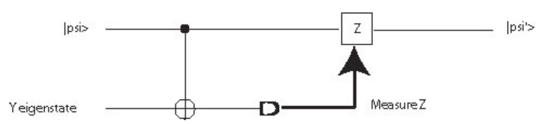

Figure 5: $|\psi'\rangle = P^{\dagger}|\psi\rangle$.

Using this sort of analysis of measurement, we can extend Knill's theorem: We have an efficient classical simulation of circuits involving Clifford group operations and also Pauli measurements and classical processing.

	IY	CNOT	ZY	Measure IZ	$\pm IZ$		
$\overline{X}$:	XI	$\rightarrow$	XX	$\longrightarrow$	YZ	$\equiv$	YI
$\overline{Z}$:	ZI		ZI	Here $M_1 = ZY$	ZI		

Table 2: Analysis of P gate construction