Steane Method (for CSS codes)

Figure 1: Note: the cat state $|0\rangle + |1\rangle \equiv |\overline{0}\rangle + |\overline{1}\rangle$.

$$
|\overline{0}\rangle + |\overline{1}\rangle = \sum_{u \in C_1} \sum_{v \in C_2} |u + v\rangle = \sum_{u \in C_1} |u\rangle
$$

$u \in C_1, C_2^\perp \subseteq C_1$.

E.g. 1 encoded block: measure all qubits. $|\overline{w}\rangle \rightarrow \sum_{w \in C_2^\perp} |u + w\rangle$. Get $u + w$ for random $w \in C_2^\perp$. Logical codewords $\leftrightarrow u \in C_1/C_2^\perp$.

$\Rightarrow$ Measurement identifies bit flip error and encoded state (in $Z$ basis).
In the figures, each line represents $n$ qubits encoded via CSS code. If there are no errors, we just get a random codeword from $C_1$. If there are bit flip errors in the data, they propagate forward along the CNOTs to the ancilla, and after measurement we get a random codeword of $C_1$ with an error in the appropriate location. Phase errors propagate backwards along the CNOTs in the second figure, and performing the Hadamard gives us a superposition of codewords from $C_2$, with bit flip errors in the locations of the phase errors from the data block.

- Repeat measurement
- Verify ancillas (to make sure we don’t have multiple bit flip errors). In the ancilla for phase errors, bit flip errors will propagate into the data. In the ancilla for bit flip errors, the Hadamard transform turns initial bit flip errors into phase errors, which can also propagate into the data.

**Ancilla Purification**

Tells us:

1.) Individual bit flip errors.
2.) Encoded bit flip errors.

Either discard bad ancillas or correct.
Measurements

How does stabilizer, $X, Z$ change under measurement?

Measure $N \in \mathcal{P}$

1.) $N \in S \Rightarrow$ nothing happens.

2.) $N \in N(S) \setminus S \Rightarrow$ measures logical Pauli in $N(S) \setminus S$.

3.) $N \notin N(S) \Rightarrow \exists M_1 \in S, \{M_1, S\} = 0$.

New stabilizer $S'$:

$\pm N \in S'$. ($+N$ if measure $+1$, $-N$ if measure $-1$)

$M_1 \notin S'$

Generators $M_2, \ldots, M_r \in S$

$[M_i, N] = 0 \Rightarrow M_i \in S'$

$\{M_i, N\} = 0 \Rightarrow M_1M_i$ commutes with $N \Rightarrow M_1M_i \in S'$

$X, Z$ etc.: $M_1X = X \Rightarrow$ choose coset representatives that commute with $N$.

$r$ generators of $S'$, $\pm N, (M_i OR M_1M_i)$.

$n - r X$'s, $n - r Z$'s.

Map $-N$ to $+N$:

Perform $M_1$ on state, commutes with $M_1$ and $X, Z$, but anticommutes with $N$, so changes the sign of $N$ without altering anything else in the new stabilizer.
Figure 4: Example: Teleportation

\[ IXX \quad \text{Measure } XXI \quad IXX \quad \text{Measure } ZZI \quad \pm ZZI \]

\[ IZZ \quad \pm XXI \quad XXI \]

\[ \overline{X}: \quad XII \rightarrow XII \rightarrow XXX \equiv IIX \]

\[ \overline{Z}: \quad ZII \quad \text{Here } M_1 = IZZ \quad ZZZ \quad \text{Here } M_1 = IXX \quad ZZZ \equiv IIZ \]

Table 1: Analysis of teleportation

Look at \( M_1 \): what do we have to do to \( M_1N \) (E.g. \( \pm XXI \)) to take it to + (i.e. \( -N \rightarrow +N \))? Do \( IZZ \) but only need to look at the 3rd qubit: \( Z \). Similarly for \( \pm ZZI \): \( IXX \) OR just \( X \).

Suppose we can do a CNOT gate but want a \( P \) gate:
CNOT \( \rightarrow P \) gate and Pauli measurements. (\( P \): \( X \rightarrow Y, Z \rightarrow Z \))

Figure 5: \( |\psi'\rangle = P^\dagger |\psi\rangle \).

Using this sort of analysis of measurement, we can extend Knill’s theorem:
We have an efficient classical simulation of circuits involving Clifford group operations and also Pauli measurements and classical processing.
$IY$  CNOT  $ZY$  Measure $IZ$  $\pm IZ$

$X$:  $XI$  $\rightarrow$  $XX$  $\rightarrow$  $YZ$  $\equiv$  $YI$

$Z$:  $ZI$  $ZI$  Here $M_1 = ZY$  $ZI$

Table 2: Analysis of $P$ gate construction