CO639 Scribe Notes

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5-qubit code	GF(4) version
$XZZXI$	$IXXXX$
$IXZZX$	$IZZZZ$
$XIXZZ$	$XIXZY$
$ZXIXZ$	$ZIZYX$

Two QECCs are equivalent iff one of them can be converted to the other via:

1. permutations of qubits
2. unitary operations on individual qubits

How does a unitary affect the code?

$$M|\overline{\psi}\rangle = M|\overline{\psi}\rangle, \forall|\overline{\psi}\rangle \in C$$

$$U|\overline{\psi}\rangle \in U(C) \Rightarrow (UMU^\dagger)U|\overline{\psi}\rangle = UM|\overline{\psi}\rangle = U|\overline{\psi}\rangle$$

So, $UMU^\dagger$ is in the new stabilizer *if* it is in $\mathcal{P}$

Two Lessons

1. $M \longmapsto UMU^\dagger$
2. Interested in U's that transform Pauli operators to Pauli operators

Def: Clifford Group $\mathcal{C} = \{U | UPU^\dagger = \mathcal{P}\}$

- $U \cdot U^\dagger$ is an automorphism of $\mathcal{P}$

- $U(PQ)U^\dagger = (UPU^\dagger)(UQU^\dagger)$
- I is in $\mathcal{C}$ (since it is a group)

Is Hadamard?

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$HXH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = Z$$

$$HZH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = X$$

$$HYH = \pm iHXZH = \pm i(HXH)(HZH) = \pm iZX = -Y$$

Is Pauli Group?

$$X(X)X^\dagger = X$$

$$X(Z)X^\dagger = -Z$$

$$X(Y)X^\dagger = -Y$$

So, for $P, Q \in \mathcal{P}$, $PQP^\dagger = \pm Q$
$+$ if [P,Q] $= 0$
$-$ if {P,Q} $= 0$

Is CNOT?

$$X \otimes I \to X \otimes X$$

$$Z \otimes I \to Z \otimes Z$$

$$I \otimes X \to I \otimes X$$

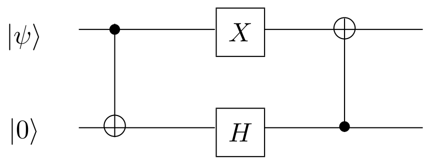
$$I \otimes Z \to Z \otimes Z$$

$P = \text{diag}(1, i)$ is also in $\mathcal{C}$. $\mathcal{C}$ is generated by $H, P, CNOT$.

In addition to stabilizer also interested in $\overline{X}, \overline{Z}$
Eigenstates of $\overline{Z}$ (Similar for $\overline{X}$),

$$\overline{Z}|\overline{\psi}\rangle = \pm|\overline{\psi}\rangle$$

$$(U\overline{Z}U^\dagger)U|\overline{\psi}\rangle = \pm U|\overline{\psi}\rangle$$
$$\Rightarrow \overline{Z} \to U\overline{Z}U^\dagger$$



$$\begin{array}{cccccccccc}
 & & \stackrel{CNOT}{\longrightarrow} & & \stackrel{X\otimes I}{\longrightarrow} & & \stackrel{I\otimes H}{\longrightarrow} & & \stackrel{CNOT}{\longrightarrow} & \\
 & I\otimes Z & \longrightarrow & Z\otimes Z & \longrightarrow & -Z\otimes Z & \longrightarrow & -Z\otimes X & \longrightarrow & Y\otimes Y \\
\overline{X} & X\otimes I & \longrightarrow & X\otimes X & \longrightarrow & X\otimes X & \longrightarrow & X\otimes Z & \longrightarrow & X\otimes Z \\
\overline{Z} & Z\otimes I & \longrightarrow & Z\otimes I & \longrightarrow & -Z\otimes I & \longrightarrow & -Z\otimes I & \longrightarrow & -Z\otimes Z
\end{array}$$

$\mathcal{C}$ preserves commutation relations

$$PQ = \pm QP$$
$$U(PQ)U^\dagger = (UPU^\dagger)(UQU^\dagger)$$
$$U(\pm 1)U^\dagger = \pm 1$$
$$(UPU^\dagger)(UQU^\dagger) = \pm(UQU^\dagger)(UPU^\dagger)$$

Suppose

$$X\otimes I = \overline{X}_1 \longrightarrow Z\otimes Z$$
$$Z\otimes I = \overline{Z}_1 \longrightarrow X\otimes I$$
$$I\otimes X = \overline{X}_2 \longrightarrow X\otimes X$$
$$I\otimes Z = \overline{Z}_2 \longrightarrow I\otimes Z$$

Then $|00\rangle$ must go to the +1-eigenvector of $\overline{Z}_1$ and $\overline{Z}_2$:

$$|00\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$$
$$|01\rangle = \overline{X}_2|00\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle$$
$$|10\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle$$

$$|11\rangle \longrightarrow -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle$$

Any transformation of $\mathcal{P}$ that preserves commutation rules and multiplication that fixes $\pm 1, \pm i$ is a Clifford group gate.
On 2n-dim binary vectors, linear map that preserves symplectic inner product.
Symplectic maps $\cong \mathcal{C}/\mathcal{P}$