Problem Set #9

Quantum Error Correction
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Due Tues., March 20, 2007

Problem #1. Entanglement-Assisted Quantum Error Correction

Suppose Alice wants to send qubits to Bob through a noisy quantum channel (assume their local quantum gates are perfect). She and Bob conveniently have some perfect pre-shared entanglement, but they have no classical communications channel. One solution is to use the quantum channel and the entanglement to send classical information, which they then can use (with additional entanglement) to teleport the qubits to Bob. However, there is a more direct route which we will explore in this problem.

a) Suppose Alice and Bob share one EPR pair. Let Alice take two qubits she wishes to transmit to Bob and her half of the EPR pair, and perform some Clifford group encoding on those three qubits, which she then sends to Bob. Bob then ends up with four qubits (his half of the EPR pair, plus Alice's three qubits). Show that there is a possible Clifford group operation for Alice so that, in the absence of noise, Bob's four qubits are in a codeword for a 4-qubit error detecting code. When there is noise, in the form of a one-qubit error in the channel, show that Bob can detect it.

   Alice sent two logical qubits to Bob via three physical qubits, and used up one EPR pair. Suppose Alice wishes to replace the EPR pairs used up in the protocol. She could do this by sending half of another EPR pair using one of her two transmitted qubits. She then has transmitted one logical qubit to Bob via three physical qubits, with one EPR pair used catalytically (i.e., it was used but replaced). Since the code detects one error, this is better than Alice and Bob could have done without a pre-shared EPR pair.

b) Suppose we have any r independent Pauli operators M_i ($r \leq 2n$) acting on n qubits. The M_is need not commute with each other. Show that there is some stabilizer (i.e., an Abelian subgroup of the Pauli group) with r generators $M_i \otimes N_i$ acting on $n + n'$ qubits. What is the smallest value of n' that will suffice? (Note that n' may depend on the particular choice of the set of M_is.)

c) Suppose that Alice and Bob share n' EPR pairs. The stabilizer code from part b encodes k' qubits. Alice is going to send n total qubits to Bob, consisting of k' data qubits, plus her halves of the n' EPR pairs, plus $n - k' - n'$ ancilla qubits which begin as $|0\rangle$ states. Show that Alice has a Clifford group operation which she may perform on her n qubits so that, in the absence of noise, Bob's $n + n'$ qubits at the end of the protocol form the stabilizer code from part b.

d) Consider the encoding from part c again. If there is noise in the channel, Bob can detect it by measuring the syndrome of the code. Recall that the noise can only act on the n qubits sent by Alice. Characterize the detectable and undetectable errors referring only to the M_is (not the N_is). If Alice uses the protocol from part c catalytically (replacing the EPR pairs used), how many logical qubits k can she send (in terms of n, r, and n')?

   Note that this problem gives us a way to convert *any* subgroup of the Pauli group, not just an Abelian subgroup, to an entanglement-assisted stabilizer-like code.

Problem #2. Two-Way Entanglement Distillation

a) Suppose we take two Werner states, produced from perfect EPR pairs $|\Phi^+\rangle$ via a depolarizing channel with total error probability p. Alice and Bob each perform a CNOT from the first noisy pair to the second, then each measures the second pair in the Z basis, and they compare results over their two-way classical communications channel. If both pairs were noiseless, the remaining state would be $|\Phi^+\rangle$, but of course in general they are not. Alice and Bob keep the first pair if the measurement results on the second pair are the same and discard it if they are not. Show that the resulting state is also a mixture of Bell states. What is the probability that they keep the pair?

b) When Alice and Bob do keep the pair, what are the three probabilities (conditional on keeping the state) that it has an X, Y, and Z error, respectively? Suppose they perform Clifford twirling after the measurement, restoring the state to a Werner state. What is the total error probability? Show that when $p < 1/2$, this procedure decreases the error rate.

c) In the case where they discard the first pair, calculate the probability of having each of an X, Y, or Z error on it, conditioned on discarding it.