Problem Set #2

Quantum Error Correction
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Due Thursday, June 9, 2016

Problem #1. Quantum Hamming bound for qudit codes
The quantum Hamming bound for qudits of dimension $p$ becomes

$$\sum_{s=0}^{t} \binom{n}{s} (p^2 - 1)^s \leq p^{n-k},$$

which must hold for non-degenerate $((n,p^k,2t+1))_p$ codes.

a) For what values of $p$ does a $[[5, 1, 3]]_p$ code saturate the quantum Hamming bound?

b) For what values of $p$ would a $[[9, 1, 5]]_p$ code saturate the quantum Hamming bound? For which values of $p$ would the code violate the quantum Hamming bound? (Note that such a code is only known to exist for prime power $p$ with $p \geq 9$.)

c) For $p = 3$, find the smallest integer values of $n$ and $k$ such that an $[[n, k, 3]]_3$ code saturates the quantum Hamming bound or show that no integer $n$ and $k$ work.

Problem #2. Logical operations for qudit code
Consider the following stabilizer code for qudits (qudits with dimension $p = 3$):

$$\begin{array}{cccc}
X & X & Z & Z \\
Z & Z & X & X \\
\end{array}$$

a) What are its parameters as a QECC?

b) Find a generating set for the logical Pauli group. (I.e., coset representatives for $\overline{X}_i$ and $\overline{Z}_i$).

c) For your choice of logical Pauli operators, write down the codeword with all logical qubits 0 expanded in the standard basis for the physical qubits.

Problem #3. Analyzing Clifford group circuits
In the following diagrams, $R = R_{\pi/4}$ is the matrix diag(1, $i$) and $H$ is the Hadamard transform.

a) For the following Clifford group circuit, compute the overall action on Paulis and use that to write down the $4 \times 4$ unitary matrix performed by the circuit:
b) For the following Clifford group circuit, use Clifford simulation techniques to compute the full probability distribution of the 8 possible classical outputs after measuring all qubits in the computational basis:

\[
\begin{align*}
|0\rangle & \quad H & \quad |0\rangle & \quad H & \\
|0\rangle & \quad R & \quad R & \quad H & \\
|0\rangle & \quad H & 
\end{align*}
\]

**Problem #4. Twirling**

Let \( S(\rho) \) be a quantum operation (a completely positive trace-preserving map) taking \( n \) qubits to \( n \) qubits. **Hint:** (For both parts) Any \( 2^n \times 2^n \) matrix can be expanded in the basis of Pauli operators.

a) Consider the following quantum operation: Choose a uniformly random \( P \in P_n / \{ \pm I, \pm iI \} \) (i.e., a Pauli ignoring global phase). Apply \( P^\dagger \), then \( S \), then \( P \) (for the same \( P \)). Show that, averaging over \( P \), the resulting quantum operation is a Pauli channel.

b) Now instead of choosing a random Pauli, choose a random Clifford and do the same thing, i.e., uniformly random \( C \in C_n / \{ e^{i\phi}I \} \), apply \( C^\dagger \), then \( S \), then \( C \). Show that, averaging over \( C \), the resulting quantum channel is a depolarizing channel.