Problem Set #3

Quantum Error Correction
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Due Thursday, June 23, 2016

Problem #1. Transversal gates for qutrit code

For this problem, consider the following $[[4,2,2]]_3$ stabilizer code on qutrits:

$$\begin{array}{c|cccc} & X & X & X^{-1} & X^{-1} \\ & Z & Z & Z & Z \\ \hline \overline{X}_1 & X & X^{-1} & I & I \\ \overline{Z}_1 & Z & I & Z & I \\ \overline{X}_2 & I & X & I & X^{-1} \\ \overline{Z}_2 & I & I & Z & Z^{-1} \end{array}$$

For each of the physical gates below, determine if it is valid gadget (meaning it preserves the codespace) and, if so, what logical gate it performs. (You can specify the logical gate via a circuit or as a transformation on the logical Paulis.)

a) $\mathcal{F}^{\otimes 4}$, where $\mathcal{F}$ is the qutrit Fourier transform, $\mathcal{F}|a\rangle = \sum_b \omega^{ab}|b\rangle$.

b) $S_2^{\otimes 4}$, where S_2 is multiplication by 2, $S_2|a\rangle = |2a\rangle$, with arithmetic mod 3.

c) $R^{\otimes 4}$, where R is the quadratic phase gate, $R|a\rangle = \omega^{a(a-1)/2}|a\rangle$.

d) $SUM^{\otimes 4}$, where SUM is the two-qutrit sum gate, $SUM|a\rangle|b\rangle = |a\rangle|a+b\rangle$, with arithmetic mod 3.

e) Find a valid gadget of the form $\mathcal{F}^{a_1} \otimes \mathcal{F}^{a_2} \otimes \mathcal{F}^{a_3} \otimes \mathcal{F}^{a_4}$, with $\mathcal{F}$ the Fourier transform as in part a and not all of the a_i's are the same. Give the logical gate performed by this transversal gate.

Problem #2. Bell state preparation gadget

For this problem, we will consider a preparation gadget for the two-qubit Bell state $|\overline{00}\rangle + |\overline{11}\rangle$. We will use a stabilizer QECC that encodes one qubit per block. (Think of the 7-qubit code if you like, but everything in this problem should be done for more general $[[n,1,2t+1]]$ codes.) For parts b and c, a full proof is not required, but you should give some explanation of the logic behind your answers.

a) Write down the Preparation Correctness Property for a Bell state preparation gadget. Also write down two versions of the Preparation Propagation Property (PPP) when there are s faults in the gadget, a weak version allowing s errors per block and a strong version allowing s errors total between the blocks.

b) Explain how to make a fault-tolerant Bell state preparation gadget using only the ideas of Shor state preparation and logical measurement. Does it satisfy the weak or strong version of the PPP?

c) Suppose you make a Bell state preparation gadget by using a circuit composed of two $|\overline{0}\rangle$ state preparation gadgets, a Hadamard gadget, and a CNOT gadget. (All are fault-tolerant gadgets.) Does it satisfy the weak or strong version of the PPP?

Problem #3. Repetition of syndrome measurement in Shor error correction

For Shor error correction and a distance 3 code, consider the following method of repeated syndrome measurement: Measure the error syndrome twice. If both syndrome measurements are the same, use that value. If the syndrome measurements differ but the first syndrome measured is 0 (corresponding to no error), deduce the trivial error. If the syndrome measurements differ but the first syndrome is non-zero, use the second syndrome to deduce the error.

a) Show that this method of repeating the syndrome and deducing the error satisfies the ECCP for a code correcting 1 error.

b) For the 7-qubit code, show that the ECRP is not satisfied by giving a combination of an error on the input state to the EC gadget and a fault during the gadget that cause the ECRP to fail.

c) For the 5-qubit code, show that the ECRP is satisfied.