On the Miracle of the Multiplication of the Loaves and Fishes

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"From where did everything come?" is an intriguing question. All of us probably have asked it, and occasionally yet wonder about it in our more reflective, not-really-expecting-an-answer moments. What we give herein are some whimsical, mathematical models of how it is possible to get something from nothing, and more particularly, how it is possible to get something from almost nothing. The whimsical element in this presentation is in analogizing between abstractions and realities. Hear a mathematical parable.

To answer the question, "From where did all of mathematics come?" one could somewhat accurately respond, "From nothing." How so? Look at figure 1, a Rudy Rucker cartoon [10, p. 40]. When we think of the notion *three* we can think of it as the empty set, neatly packaged. That is, identify 0 with $\{\}$, 1 with $\{\{\}\}$, 2 with $\{\{\},\{\{\}\}\}$, 3 with the thought bubble of figure 1, and so on, getting the set of all nonnegative integers, wherein each integer is the collection of all preceding integers. Thinking of opposites and ordered pairs along with an equivalence relation gives the negative integers and rational numbers, respectively. Hypothesizing a least upper bound property yields the real numbers. Relating elements of various sets of numbers give operators and structures of wonderful complexity, which in turn can be related, giving all of mathematics. That is, any part of mathematics can be thought of as nothing, neatly packaged, or more precisely, "as packaging, neatly packaged," [8] since the empty set is simply a package of nothing.

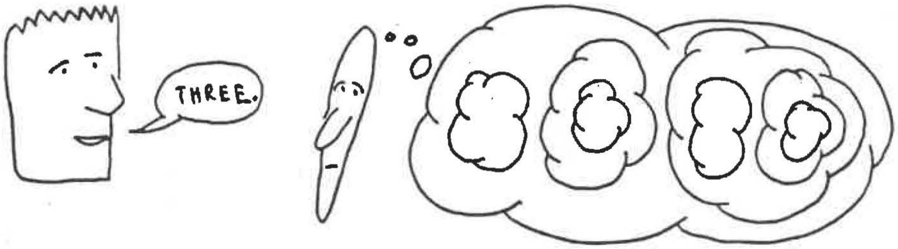

Figure 1. A representation of three.

Now what about reality? Look at this page, and zoom to the molecular level, on past the atomic structure, and below the subatomic level; space is almost all there is, so much so that—a sneaking suspicion wells up within—one wonders whether space is *all* there is. That is, perhaps all of matter is space, neatly packaged! Recent theories of physics echo this "empty" idea.

- Virtual particles are said to come "out of nothing", momentarily violating the conservation of energy principle, so as to provide for the idea of exchange particles which somehow explains the existence of the four fundamental forces of nature: gravity, electro-magnetic, nuclear, and sub-nuclear forces.
- Dirac suggested the idea that there are negative energy particles throughout space, packaged somewhat like a vast, oceanic honeycomb, which when "liberated" to positive energies could be observed either as an electron or as a "hole".
- Black holes have zero physical extent yet huge influence over great distances.
- The universe may very well be multidimensional, perhaps up to dimension 20, but whose dimensions may be folded up as a string into tiny bundles.
- The fundamental building blocks of nature wane ever smaller. Yesterday it was the proton and electron. Today it's the various quarks: *up, down, charm, strange, bottom,* and *top.* Perhaps tomorrow we'll see the quark decomposed into yet finer particles, until it bottoms out into emptiness(?).

However this preceding analogy has exhausted all that we can say regarding the question, "How can something come from nothing?" Therefore we turn to an easier question, "How can something come from *almost* nothing?" Figure 2 [4, p. 184], is a Gustave Doré cut of the Biblical miracle of the feeding of thousands of people from a few loaves of bread and small fishes, with baskets full of leftovers. We accordingly ask, "How can we *instantaneously* get, for example, two fish from one, with no sleight of hand allowed?"

Figure 2. The multiplication of the loaves and fishes.

Two approaches come to mind, namely stretching the fish or partitioning the fish, as illustrated in figure 3.

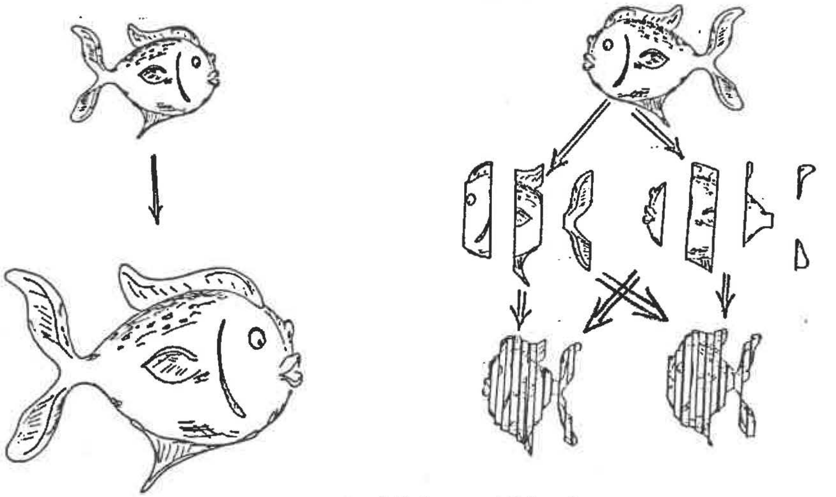

Figure 3. Fish partitioning.

Consider the first approach. For simplicity we can think of a one dimensional *fish* as a line segment $I = [0,1]$, the interval from 0 to 1. Stretch I using the function $f(x) = 3x$, so that $f(I) = [0,3]$. Remove the middle third, leaving two identical fish, namely $[0,1]$ and $[2,3]$. This process can be iterated, and generalized to three dimensions, allowing the generation of as many fish as needed. However this model has some problems in that everything is stretched linearly so that the atomic structure of the fish is now that of Brobdingnagian-like proportions in some fantasy world visited by Gulliver. If a nonlinear stretching function f is used instead so that atomic structure remains intact, then no increase of mass occurs, and f will surely shred fish fibres so that nothing recognizeable as food remains.

The second approach has more promise, and champions the dictum that the sum of the parts is greater than the whole. A classic problem along these lines is the series S, where

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$$

Grouping the terms of this series successively two at a time suggests that $S = 0$. Letting the first term stand alone, and grouping the remaining terms successively two at a time suggests that $S = 1$. Leibnitz apparently thought that $S = \frac{1}{2}$ because the above equation can be written as $S = 1 - S$, [1, p. 60]. Furthermore, for any convergent series which fails to converge absolutely, the terms of S can be rearranged so that the sum exceeds any prearranged value. To implement fish duplication using this idea however is hopeless, because of the initial need for an infinite supply of regular fish along with some kind of antimatter fish, the sum total mass of which is zero.

The straight forward cut and reassemble process as indicated in figure 3*b* will indeed double the fish supply—but also halve the mass of each fish. A more clever example of this process is given by the Vanishing Leprechaun puzzle below. Start with 15 Leprechauns and then rearrange the 3 pieces to get 14 Leprechauns. Of course the area of these Leprechauns as a group remains constant.

Figure 4. The Vanishing Leprechaun.

In 1924, Banach and Tarski came up with a more exotic cut and reassemble process—with no stretching—allowing the generation of two exact copy fish from one fish. A highly readable proof of their result is [5], wherein is also a delightful succession of 3 snapshots, showing the author wielding a huge scalpel, "successfully" performing Banach-Tarski surgery on an orange, transforming one into two.

But rather than look at the Banach-Tarski algorithm, which involves at least a five part partition and isometric movements of the parts (and since their algorithm is only valid in dimension 3 or more) [14, p. 40], it will be sufficient for our purposes to look at sets of points resembling the *pieces* of their partition.

First of all, recall what we mean by length, area, and volume. The most successful definition was given by Lebesgue at the turn of this century, who defined the *length* or *outer measure* of any set A of real numbers, denoted $m(A)$, to be the infimum of the measure of open sets U containing A, where $m(U)$ is the sum of the lengths of the intervals comprising U. Thus the measure of an interval is its length, namely, the difference in its endpoints; and the measure of A is translation invariant, which simply means that the measure of a fish is independent of where it is placed. The measure of a two or three dimensional set of ordered pairs or triples is analogously defined.

In his high hopes for this definition, Lebesgue thought that for any other set B of real numbers, it would turn out that the measure of the part of B which is A plus the measure of the part of B which is not A would sum to the measure of B. To see this last statement using diagrams, think of B as being a window, behind which is placed fish A. What is seen is the part of the fish behind the window, namely $A \cap B$, and the background in the window, namely $A' \cap B$, as in figure 5. However in 1905 [2], Vitali showed that there are fish A such that $m(A' \cap B) + m(A \cap B) > m(B)$ for some B's. These are the kind of fish—the *nonmeasurable* sets—with which we shall deal.

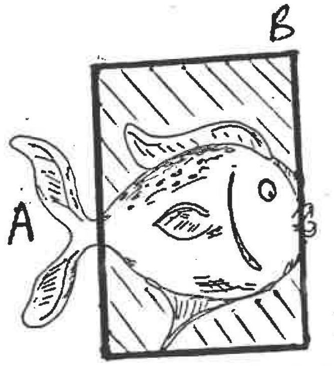

Figure 5. A fish and its background.

In particular, define A to be an *Archimedean* set of numbers if the set of all real numbers r such that $A + r = A$ is dense in R. With a bit of work, it turns out that if A is an Archimedean set with $m(A) > 0$ then $m(A \cap J) = m(J)$ for all intervals J on the real line, [12].

Here is a good example of these Archimedean sets which we shall call *Ungar's fish partition,* [12]. Define an equivalence relation on the set of real numbers so that x and y are related if and only if $x - y = \frac{p}{q}$ where p and q are integers with q odd. From each equivalence class choose α. Let A be the set of all numbers of the form $\alpha + \frac{p}{q}$ where both p and q are odd. It follows that A' is the set of all numbers of the form $\alpha + \frac{2p}{q}$ where q is odd and p is an integer. Furthermore, A is Archimedean since $A + \frac{2p}{q} = A$ for any integer p and any odd integer q. Since $A' + \frac{p}{q} = A$ it then follows that $m(A \cap I) = 1 = m(A' \cap I)$, giving us two fish with length 1 from a fish of length 1.

Before proceeding with the multiplication of more fish, let us consider some natural questions regarding this partition.

- Are each of Ungar's two fish as satisfying as the original? Are the masses of these fish the same as the original, or perhaps half of the original? Have we destroyed the very fabric of matter in this partition—a partition which sends parts of every bit into two widely separated places? These questions, we can not answer.
- Is the Lebesgue measuring stick somehow faulty in that it can not detect the *holes* in Ungar's fish? In 1936, Marczewski [2] showed that if we want a measuring stick so that the measure of an interval is its length and so that the measure is translation invariant, then there will always be these measure defying fish partitions. So the answer to this question is that there is no better measuring stick than the one we have used.
- But is there some measuring stick showing that each of Ungar's fish is half the original? Mabry [8] showed that there are length preserving, translation invariant (but only finitely additive) measure extensions m^* on the set of subsets of R such

that $m^*(A \cap J) = \frac{m^*(J)}{2}$ for some sets A and for all intervals J, a property which Lebesgue's measuring stick fails to have. So the answer to this question is yes. But there will be measure defying fish partitions for these kinds of measures as well, in the sense that there will always be sets for which m^* is not translation invariant.

- Is the Banach-Tarski partition better than Ungar's partition? In some ways, the answer is yes because the reassembling of the pieces of the Banach-Tarski partition creates exact duplicates, with no holes. But in other ways the answer is no because the pieces of the Banach-Tarski partition are in the same class as Ungar's fish.

To multiply more fish what can we do? Kellerer [7] gives an example of partitioning I into a countable number of disjoint sets all of which have measure 1, which means that we can generate all the fish we want this way. Stromberg [13] and Sierpinski & Lusin [11] go even further, demonstrating how to partition I into an uncountable number of disjoint sets all of which have measure 1. These methods generalize of course to three dimensional fish. Such an algorithm fills many baskets indeed!

Two related problems.

Besides the classic Banach-Tarski partition, here are two other classic problems involving nonmeasurable sets.

- *Squaring the circle.* Rather than construct a square of equal area to a given circle using only compass and straight-edge, this variant of Tarski's asks

  > *Is it possible to partition the circle into a finite number of pieces so that some translations of the pieces form a partition of a square of equal area?*

  Miklós Laczkovich (1988) answers "yes" [6], albeit the pieces of the partition are nonmeasurable.

- *The continuous four color problem.* An old problem of Erdös asks

> *Is is possible to color the points of the plane using four colors so that all points one unit apart are colored differently?*

It's an easy, fun exercise to see that three colors are insufficient, and a bit more difficult to show that seven colors are sufficient, (see [3] for these solutions). The current thinking (see [15] for example) is that the puzzle's answer is "yes" but that the sets of each color are nonmeasurable.

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