Finding Explicit Solutions to Diophantine Equations

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1 Introduction

Diophantine equations are equations with integer coefficients constructed using addition, subtraction, multiplication, and exponentiation. The only solutions of interest to Diophantine equations are typically natural number solutions. Since Martin Davis, Julia Robinson, Hilary Putnam, and Yuri Matijasevi proved that any recursively enumerable set, i.e. sets that can be described algorithmically, is Diophantine [2] [5], many special sets of numbers have been represented by Diophantine equations. Notably, James Jones et al. have created a set of Diophantine equations to represent the set of prime numbers [4]. Jones et al. managed to create a polynomial of degree 25 in 26 variables from the set of equations whose positive values are strictly prime. The aforementioned polynomial:

$$\begin{aligned}&(k+2)\{1-[wz+h+j-q]^2-[(gk+2g+k+1)\cdot(h+j)+h-z]^2-[2n+p+q+z-e]^2\\&-[16(k+1)^3\cdot(k+2)\cdot(n+1)^2+1-f^2]^2-[e^3\cdot(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2\\&-[16r^2y^4(a^2-1)+1-u^2]^2-[((a+u^2(u^2-a))^2-1)\cdot(n+4dy)^2+1-(x+cu)^2]^2-[n+l+v-y]^2\\&-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2\\&-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2\}\end{aligned}$$

The Diophantine representation of the primes heavily relies on the Diophantine representation of the exponential function. Davis showed in 1973 that the exponential function, $n^k = m$, can be represented in 12 equations which are only true when $n^k = m$ is also true [1]. These equations use the properties of the Pell equation

$$x^2 - dy^2 = 1$$

in order to force specific integer solutions.

While the theoretical side of these Diophantine representations has been mapped out, little has been done in terms of explicit solutions. Nachiketa Gupta attempted to formulate explicit solutions to the prime representing Diophantine equations for the prime number 2 but was unsuccessful in actually finding the whole solution set [3]. Of the total 26 variables, Gupta found values for only 22 of them. The problem he ran into regarded the size of the solutions of the last four variables. That is, some equations require numbers so large that it is impossible to write them down, let alone calculate them.

In this paper, we will explore the root of the problem—the exponential function—and show that it is impossible to find solutions for the equations representing the function when $k \geq 5$. We will first find general explicit solutions to the equations in [1] and then use these to find actual explicit solutions and prove the upper bound on k.

2 General Explicit Solutions

Before finding general solutions to the Diophantine equations representing the exponential function we need some properties of the special Pell equation where

$$x^2 - (a^2 - 1)y^2 = 1, \quad \text{for} \quad a > 1. \tag{1}$$

The trivial solutions to this equation are

$$x = 1, y = 0$$
$$x = a, y = 1.$$

Definition 2.1. $x_n(a) + y_n(a)\sqrt{a^2 - 1} = (a + \sqrt{a^2 - 1})^n$

For simplicity, $x_n(a)$ and $y_n(a)$ will be written as x_n and y_n where dependence on a is implied.

Lemma 2.1. x_n and y_n are solutions to (1)

Lemma 2.2. $x_{n+1} = 2ax_n - x_{n-1}$ and $y_{n+1} = 2ay_n - y_{n-1}$

Lemma 2.3. When n is even, y_n is even and when n is odd, y_n is odd.

The proofs to these lemmas can be found in [1]. On top of these lemmas we will need the following:

Lemma 2.4. If n is even, x_n is odd. If n is odd and a is odd, $x_n(a)$ is odd and if n is odd and a is even, $x_n(a)$ is even.

Proof. The lemma holds for $x_o = 1$, $x_1 = a$. Following inductively:

$$x_{n+1} = 2ax_n - x_{n-1}$$
$$x_{n+1} \equiv -x_{n-1} \equiv x_{n-1} \pmod 2$$

□

Note that this result is also deducible from (1) and 2.3. When y or $a^2 - 1$ is even x must be odd in order to have a difference of 1. Both y and $a^2 - 1$ being odd force x to be even.

Since solutions to (1) are recursive we may use a characteristic equation to find general solutions to x and y.

$$x_{n+1} - 2ax_n + x_{n-1} = 0$$
$$r^{n+1} - 2ar^n + r^{n-1} = 0$$
$$r^2 - 2ar + 1 = 0$$

$$r = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$$
$$a_n = (a + \sqrt{a^2 - 1})^n, \quad b_n = (a - \sqrt{a^2 - 1})^n$$

$$c_n = 2ac_{n-1} - c_{n-2} = k_1 2a(a + \sqrt{a^2 - 1})^n - k_2(a - \sqrt{a^2 - 1})^n$$

Using $c_0 = 0$, $c_1 = 1$ for y and $c_0 = 1$, $c_1 = a$ for x and solving for k_1 and k_2 we have:

$$y_n = \frac{1}{2\sqrt{a^2-1}}\left((a+\sqrt{a^2-1})^n - (a-\sqrt{a^2-1})^n\right)$$
$$x_n = \frac{1}{2}\left((a+\sqrt{a^2-1})^n + (a-\sqrt{a^2-1})^n\right)$$

With this knowledge, we are ready to start writing the nth solution to the special Pell equation in terms of n and a. We will follow the method outlined in [1] to find a general explicit solution to equations derived by Davis. The aforementioned equations

$$x^2 - (a^2-1)y^2 = 1 \tag{2}$$

$$u^2 - (a^2-1)v^2 = 1 \tag{3}$$

$$s^2 - (b^2-1)t^2 = 1 \tag{4}$$

$$v = ry^2 \tag{5}$$

$$b = 1 + 4py = a + qu \tag{6}$$

$$s = x + cu \tag{7}$$

$$t = k + 4(d-1)y \tag{8}$$

$$y = k + e - 1 \tag{9}$$

$$(x - y(a-n) - m)^2 = (f-1)^2(2an - n^2 - 1)^2 \tag{10}$$

$$m + g = 2an - n^2 - 1 \tag{11}$$

$$w = n + h = k + l \tag{12}$$

$$a^2 - (w^2-1)(w-1)^2z^2 = 1 \tag{13}$$

Theorem 2.5. *$m = n^k$ if and only if (2)-(13) have solutions in the remaining arguments.*

We will concern ourselves with the case $n \leq k$ for simplicity, although general solutions are just as easily found for $n > k$. The only equation that features both n and k is (12), so that is where we will start.

Equation (12) is just saying that $w > n$ and $w > k$. In order to keep solutions as minimal as possible we have

$$w = n + h = k + 1, \quad h = k + 1 - n, \quad l = 1. \tag{14}$$

Equation (13) slightly deviates from the typical special Pell equation as it expresses the y^2 in (1) as the product of two squares. Namely, $(w-1)^2z^2$. This is done so that a is large enough to satisfy (10) and (11). As we will see later in the paper we will use the k the solution to the majority of these Pell equations. Accordingly, we have

$$a = x_{w-1}(w) \quad \text{and} \quad z = \frac{y_{w-1}(w)}{w-1}.$$

This can be written as

$$a = \frac{1}{2}\left(\left(k+1+\sqrt{(k+1)^2-1}\right)^k + \left(k+1-\sqrt{(k+1)^2-1}\right)^k\right) \tag{15}$$

$$z = \frac{1}{2k\sqrt{(k+1)^2-1}}\left(\left(k+1+\sqrt{(k+1)^2-1}\right)^k - \left(k+1-\sqrt{(k+1)^2-1}\right)^k\right). \tag{16}$$

The next equation to tackle is (2) which encodes the central property that assures (2.5) is true. Equations (3)-(9) are simply there to ensure $x = x_k(a)$ and $y = y_k(a)$. The solutions are

$$x = x_k = \frac{1}{2}\left(\left(a+\sqrt{a^2-1}\right)^k + \left(a-\sqrt{a^2-1}\right)^k\right) \tag{17}$$

$$y = y_k = \frac{1}{2\sqrt{a^2-1}}\left(\left(a+\sqrt{a^2-1}\right)^k - \left(a-\sqrt{a^2-1}\right)^k\right). \tag{18}$$

These solutions are written in terms of a and k as writing in just terms of k would create an excessively complicated expression. We could at any point plug our solution to a back into these solutions in order to have something just in terms of k.

(5) guarantees that $u = x_{iky_k}$ and $v = y_{iky_k}$ for some i in (3). We will use $i = 2$ to ensure that iky_k is even, in accordance with Davis's instructions. To satisfy both (3) and (5) we have:

$$u = x_{2ky} = \frac{1}{2}\left(\left(a+\sqrt{a^2-1}\right)^{2ky} + \left(a-\sqrt{a^2-1}\right)^{2ky}\right) \tag{19}$$

$$v = y_{2ky} = \frac{1}{2\sqrt{a^2-1}}\left(\left(a+\sqrt{a^2-1}\right)^{2ky} - \left(a-\sqrt{a^2-1}\right)^{2ky}\right) r = \frac{v}{y^2}. \tag{20}$$

At this point, we will have to deviate from the method outlined by Davis as it is inadequate for finding general solutions. He proposed using the Chinese Remainder Theorem to solve (6), however, that requires knowing certain inverses that we do not know in this case.

Lemma 2.6. When k is even, $x_{2ky_k}(a) \equiv 1 \pmod{4y_k}$. When k is odd and a is odd, $x_{2ky_k}(a) \equiv 1 \pmod{4y_k}$ and when k is odd and a is even, $x_{2ky_k} \equiv 2y_k + 1 \pmod{4y_k}$.

Proof. Starting with the case where k is even, y is even by Lemma 2.3.

$$x_{2ky_k}(a) + y_{2ky_k}(a)\sqrt{a^2-1} = \left(\left(a+\sqrt{a^2-1}\right)^{2k}\right)^{y_k} \tag{21}$$

Expanding the right-hand side with the binomial theorem and collecting the components without $\sqrt{a^2-1}$ we have

$$x_{2ky_k} = \sum_{i=0}^{\frac{y_k}{2}} \binom{y_k}{2i} \cdot x_{2k}^{y_k-2i} \cdot \left(y_{2k}^2\left(a^2-1\right)\right)^i$$

$$x_{2ky_k} \equiv (x_{2k})^{y_k} \pmod{y_k}$$

$y_k = 2m$ for some m since y_k is even and $x_n^2 \equiv 1 \pmod{y_n}$ by equation (1). Combining these two with the equivalence above we have:

$$\begin{aligned} x_{2ky_k} &\equiv \left((x_{2k})^2\right)^m \pmod{y_k} \\ &\equiv 1^m = 1 \pmod{y_k} \end{aligned}$$

By Lemma (2.4), x_k is odd and so can be represented as $2n+1$ for some n. Since $y_k^2 \equiv 0$,

$$\begin{aligned} x_{2ky_k} &\equiv x_k^{y_k} \pmod{4} \\ &\equiv (2n+1)^{2m} \pmod{4} \\ &\equiv \left(4n^2+4n+1\right)^m \pmod{4} \\ &\equiv 1^m = 1 \pmod{4}. \end{aligned}$$

Since $x_{2ky_k} \equiv 1 \pmod{y_k}$ and $x_{2ky_k} \equiv 1 (\bmod 4), x_{2ky} \equiv 1 \pmod{4y_k}$.

Continuing on to the case where k is odd, y_k is odd. Rearranging components from (22) in a slightly different way we have

$$x_{2ky_k} = \sum_{i=0}^{y_k} \binom{2y_k}{2i} \cdot x_k^{2y_k-2i} \cdot \left(y_k^2\left(a^2-1\right)\right)^i.$$

And so by the same logic from earlier in the proof:

$$x_{2ky_k} \equiv 1 \pmod{y_k}$$

Going by cases, if a is odd, x_k is also odd.

$$\begin{aligned} a^2 - 1 &\equiv 0 \pmod{4} \\ x_k^2 &\equiv 1 \pmod{4} \end{aligned}$$

Then:

$$\begin{aligned} x_{2ky_k} &\equiv x_k^{2y_k} \pmod{4} \\ &\equiv 1^{y_k} = 1 \pmod{4}. \end{aligned}$$

And so $x_{2ky_k} \equiv 1 \pmod{4y_k}$ when k is odd and a is even. If a is even, x_k is also even.

$$x_k^2 \equiv 0 \pmod{4}$$

So:

$$\begin{aligned} x_{2ky_k} &\equiv \left(y^2\left(a^2-1\right)\right)^{y_k} \pmod{4} \\ &\equiv \left(a^2-1\right)^{y_k} \pmod{4} \\ &\equiv (-1)^{y_k} = -1 \pmod{4} \end{aligned}$$

$x_{2ky_k} \equiv 1 \,(\text{mod} y_k)$ means $u = 1 + my_k$ for some m and $x_{2ky_k} \equiv -1 (\text{mod} 4)$ means $ky \equiv 2$ (mod 4). Since y_k is odd, any k of the form $2 + 4n$ for some n satisfies the congruence. Therefore

$$u = 1 + (2 + 4n)y_k = 1 + 2y_k + 4y_k \equiv 1 + 2y_k \pmod{4y_k}.$$

□

From here the general solutions to (2)-(13) change depending on whether k is even or odd. We will start with the case where k is even. By Lemma 2.6:

$$u = 4yi + 1, \quad \text{for some } i.$$
$$i = \frac{u-1}{4y}$$

Some algebra:

$$\begin{aligned}
4yi &= u - 1 \\
4y(i(4y - a + 1)) &= u(4y - a + 1) - (4y - a + 1) \\
4y(i(4y - a + 1)) + 4y &= u(4y - a + 1) - (4y - a + 1) + 4y \\
4y(i(4y - a + 1) + 1) &= u(4y - a + 1) + a - 1 \\
1 + 4y(i(4y - a + 1) + 1) &= a + u(4y - a + 1)
\end{aligned}$$

Now we have something of the form $1 + 4yp = a + uq$. We can now solve equation (7).

$$\begin{aligned}
p &= \frac{u-1}{4y}(4y - a + 1) + 1 \\
q &= 4y - a + 1
\end{aligned}$$

Now for the case where k is odd. By Lemma 2.6:

$$u = 4yj + 2y + 1, \quad \text{for some } j.$$
$$j = \frac{u - 2y - 1}{4y}$$

Some more algebra:

$$\begin{aligned}
4yj &= u - 2y - 1 \\
4y(j(2y - a + 1)) &= (u - 2y - 1)(2y - a + 1) \\
4y(j(2y - a + 1)) + 4y\left(y - \frac{1}{2}a + 1\right) &= (u - 2y - 1)(2y - a + 1) + 4y\left(y - \frac{1}{2}a + 1\right) \\
4y\left(j(2y - a + 1) + y - \frac{1}{2}a + 1\right) &= u(2y - a + 1) - 2y(2y - a + 1) \\
&\quad - (2y - a + 1) + 4y^2 - 2ya + 4y \\
&= u(2y - a + 1) - 4y^2 + 2ya - 2y - 2y \\
&\quad + a - 1 + 4y^2 - 2ya + 4y \\
&= u\,(2y - a + 1) + a - 1. \\
1 + 4y\left(j(2y - a + 1) + y - \frac{1}{2}a + 1\right) &= a + u(2y - a + 1)
\end{aligned}$$

So:

$$p = \frac{u-2y-1}{4y}(2y-a+1) + y - \frac{1}{2}a + 1$$
$$q = 2y - a + 1$$

Use the corresponding p and q depending on the parity of k to determine b from (6). Equation (6) is somewhat convoluted in nature to ensure that $b \equiv 1(\text{mod}4y), b \equiv a(\text{mod}u), b > 4y$, and $b > u$. This has relevance in ensuring (4), (7), and (8) are all simultaneously true.

Now we need to find s and t in (4). Solutions:

$$s = x_k(b)$$
$$t = y_k(b)$$

And so:

$$s = \frac{1}{2}\left(\left(b+\sqrt{b^2-1}\right)^k + \left(b-\sqrt{b^2-1}\right)^k\right)$$
$$t = \frac{1}{2\sqrt{b^2-1}}\left(\left(b+\sqrt{b^2-1}\right)^k - \left(b-\sqrt{b^2-1}\right)^k\right).$$

The equations that remain are (8)-(12). Equations (7) and (8) perform similar jobs causing $c \equiv x(\text{mod}u)$ with $c > u$ and $t \equiv k(\text{mod}4y)$ with $t >= k$ respectively. Equation (10) helps to ensure $x = x_k$ and $y = y_k$. Without it, it's possible to always have $x = x_1$ and $y = y_1$. For (8)-(10):

$$c = \frac{s-x}{u}$$
$$d = 1 + \frac{t-k}{4y}$$
$$e = 1 + y - k$$

It can be shown that [1] that c, d, and e will be integers because of the choices we made for the previous variables, even though it appears they may not be.

Equations (10) and (11) make sure that we've chosen the right m to satisfy $n^k = m$. Equation (10) ensures that $x - y(a-n) - m$ is some integer multiple of $2an - n^2 - 1$. In other words: $x - y(a-n) \equiv m \left(\text{mod}2an - n^2 - 1\right)$. Equation (11) forces $m < 2an - n^2 - 1$. These equations can be resolved similarly to those above:

$$f = 1 + \frac{x - y(a-n) - m}{2an - n^2 - 1}$$
$$g = 2an - n^2 - 1 - m$$

3 Explicit Solutions

Now that we have general solutions to all of the variables in Davis's equations we can apply them to actual cases of $n^k = m, k \geq n$. We will start with showing $1^1 = 1$.

$$n = 1, \quad k = 1, \quad m = 1$$

By (14)-(29):

$$
\begin{aligned}
&w = 1+1 = 2, \quad h = 1+1-1 = 1 \\
&a = \frac{1}{2}\left(\left(1+1+\sqrt{(1+1)^2-1}\right)^1 + \left(1+1-\sqrt{(1+1)^2-1}\right)^1\right) \\
&= \frac{1}{2}(2+\sqrt{3}+2-\sqrt{3}) \\
&= 2 \\
&z = \frac{1}{2\sqrt{(1+1)^2-1}}\left(\left(1+1+\sqrt{(1+1)^2-1}\right)^1 - \left(1+1-\sqrt{(1+1)^2-1}\right)^1\right) \\
&= \frac{1}{2\sqrt{3}}((1+1+\sqrt{3})-(1+1-\sqrt{3})) \\
&= 1.
\end{aligned}
$$

Verifying with (13):

$$
\begin{gathered}
2^2 - \left(2^2-1\right)(2-1)^2 1^2 = 1 \\
4-(3)(1) = 1 \\
1 = 1
\end{gathered}
$$

For x and y :

$$
\begin{aligned}
x &= \frac{1}{2}\left(\left(2+\sqrt{2^2-1}\right)^1 + \left(2-\sqrt{2^2-1}\right)^1\right) \\
&= \frac{1}{2}((2+\sqrt{3})+(2-\sqrt{3})) \\
&= 2 \\
y &= \frac{1}{2\sqrt{2^2-1}}\left(\left(2+\sqrt{2^2-1}\right)^1 - \left(2-\sqrt{2^2-1}\right)^1\right) \\
&= \frac{1}{2\sqrt{3}}((2+\sqrt{3})-(2-\sqrt{3})) \\
&= 1.
\end{aligned}
$$

Plugging into (2):

$$
\begin{gathered}
2^2 - \left(2^2-1\right)1^2 = 1 \\
1 = 1
\end{gathered}
$$

Continuing on:

$$
\begin{aligned}
u &= \frac{1}{2}\left(\left(2+\sqrt{2^2-1}\right)^{2(1)(1)} + \left(2-\sqrt{2^2-1}\right)^{2(1)(1)}\right) \\
&= \frac{1}{2}\left((2+\sqrt{3})^2 + (2-\sqrt{3})^2\right) \\
&= \frac{1}{2}(4+4\sqrt{3}+3+4-4\sqrt{3}+3) \\
&= 7 \\
v &= \frac{1}{2\sqrt{2^2-1}}\left(\left(2+\sqrt{2^2-1}\right)^{2(1)(1)} - \left(2-\sqrt{2^2-1}\right)^{2(1)(1)}\right) \\
&= \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})^2 - (2-\sqrt{3})^2\right) \\
&= \frac{1}{2\sqrt{3}}(4+4\sqrt{3}+3-4+4\sqrt{3}-3) \\
&= 4 \\
r &= \frac{4}{1^2} \\
&= 4
\end{aligned}
$$

Again verifying with (3) and (5):

$$
\begin{aligned}
7^2 - \left(2^2-1\right)4^2 &= 1 \\
49 - (3)16 &= 1 \\
1 &= 1 \\
4 &= 4(1)
\end{aligned}
$$

Moving on to $p, q,$ and b :

$$
\begin{aligned}
p &= \frac{7-2(1)-1}{4(1)}(2(1)-2+1)+1-\frac{1}{2}(2)+1 \\
&= \frac{4}{4}(1)+1 \\
&= 2 \\
q &= 2(1)-2+1 \\
&= 1 \\
b &= 1+4(1)(2) = 2+(7)(1) \\
&= 9
\end{aligned}
$$

In finding b we also ensured that p and q were correct. For the final Pell equation:

$$
\begin{aligned}
s &= \frac{1}{2}\left(\left(9+\sqrt{9^2-1}\right)^1+\left(9-\sqrt{9^2-1}\right)^1\right)\\
&= \frac{1}{2}((9+\sqrt{80})+(9-\sqrt{80}))\\
&= 9\\
t &= \frac{1}{2\sqrt{9^2-1}}\left(\left(9+\sqrt{9^2-1}\right)^1-\left(9-\sqrt{9^2-1}\right)^1\right)\\
&= \frac{1}{2\sqrt{80}}((9+\sqrt{80})-(9-\sqrt{80}))\\
&= 1
\end{aligned}
$$

To check:

$$
\begin{aligned}
9^2-\left(9^2-1\right)1^2 &= 1\\
1 &= 1
\end{aligned}
$$

For the values of c, d, e :

$$
\begin{aligned}
c &= \frac{9-2}{7}\\
&= 1\\
d &= 1+\frac{1-1}{4(1)}\\
&= 1\\
e &= 1+1-1\\
&= 1
\end{aligned}
$$

Now we will find f :

$$
\begin{aligned}
f &= 1+\frac{2-1(2-1)-1}{2(2)(1)-1^2-1}\\
&= 1+\frac{0}{4-1-1}\\
&= 1
\end{aligned}
$$

And for g :

$$
\begin{aligned}
g &= 2(2)(1)-1-1-1\\
&= 1
\end{aligned}
$$

This concludes the derivation of the solutions to Davis's equations that describe $1^1 = 1$. This can be done for any n, k, and m such that $n^k = m$, however, as we will discuss in the next section, there is a limitation to finding solutions for all n, k, and m. Solutions to $2^2 = 4$ can be found in Appendix.A.

4 Explicit Solutions Limitation

Here we show that there is a point where we can no longer find solutions to the equations representing the exponential function. The variable with the largest value in (**??**)-(**??**) is s, so we will focus on finding the size of s for a given $n^k = m$. Starting with $n = 5, k = 5$, and $m = 3125$ and ignoring certain variables that don't have bearing on s:

$$\begin{aligned}
w &= 5 + 1 = 6 \\
a &= \frac{1}{2}\left((5 + 1 + \sqrt{(5+1)^2 - 1})^5 + (5 + 1 - \sqrt{(5+1)^2 - 1})^5\right) \\
&= 120126 \\
y &= \frac{1}{2}\left((120126 + \sqrt{120126^2 - 1})^5 - (120126 - \sqrt{120126^2 - 1})^5\right) \\
&\approx 3 \cdot 10^{21}
\end{aligned}$$

We will start using approximations here as the numbers start to get substantially larger. we will use the fact that $n + \sqrt{n^2 - 1} \approx 2n$ and $n - \sqrt{n^2 - 1} \approx 0$ for sufficiently large n.

$$\begin{aligned}
u &\approx \frac{1}{2}\left((120126 + \sqrt{120126^2 - 1})^{2(5)(3 \cdot 10^{21})} + (120126 - \sqrt{120126^2 - 1})^{2(5)(3 \cdot 10^{21})}\right) \\
&\approx \frac{1}{2}(2 \cdot 120126)^{2(5)(3 \cdot 10^{21})} \\
&\approx 10^{\log_{10}(\frac{1}{2}) + 2(5)(3 \cdot 10^{21}) \cdot \log_{10}(2 \cdot 120126)} \\
&\approx 10^{1.6 \cdot 10^{23}}
\end{aligned}$$

We will just use q to calculate b.

$$\begin{aligned}
q &\approx 2 \cdot 3 \cdot 10^{21} - 120126 + 1 \\
&\approx 6 \cdot 10^{21}
\end{aligned}$$

$$\begin{aligned}
b &\approx 120126 + 10^{1.6 \cdot 10^{23}} \cdot 3 \cdot 10^{21} \\
&\approx 3 \cdot 10^{1.6 \cdot 10^{23}}
\end{aligned}$$

For s:

$$\begin{aligned}
s &\approx \frac{1}{2}\left((3 \cdot 10^{1.6 \cdot 10^{23}} + \sqrt{(3 \cdot 10^{1.6 \cdot 10^{23}})^2 - 1})^5 + (3 \cdot 10^{1.6 \cdot 10^{23}} - \sqrt{(3 \cdot 10^{1.6 \cdot 10^{23}})^2 - 1})^5\right) \\
&\approx \frac{1}{2}(2 \cdot 3 \cdot 10^{1.6 \cdot 10^{23} + 21})^5 \\
&\approx 10^{log_{10}\frac{1}{2} + 5 \cdot log_{10}(10^{1.6 \cdot 10^{23} + 21})} \\
&\approx 10^{8 \cdot 10^{23}}
\end{aligned}$$

Both s and u have a number of digits in the realm of 10^{23} digits. This is far beyond the number of digits producible by even the best supercomputers, let alone computable. It would require around $2 \cdot 10^{19}$ pages completely filled with digits in order to write down s. It would be physically impossible to write these numbers down, so in a sense, they're completely useless. This will be true for $k \geq 5$.

The nature of Davis's equations imposes the solution of (2) to be x_k. Therefore, there are not the same constraints on n as there are on k. The value of n is allowed to be sufficiently large if k is sufficiently small. The value of m does not have as powerful an effect on the size of solutions to (2)-(13) and so is not restraining the solutions like k and n do.

This has implications for all other sets of Diophantine equations that require the use of the exponential functions. The reason Gupta [3] could not calculate values to the equations in [4] is that they require the use of the exponential function for some $k \geq 5$.

Appendices

A Solutions to 2^2

Even the solutions to Davis's equations for $2^2 = 4$ are extremely large as can be seen below. The problem again arises from the values of u and v which are immensely larger than x and y and cause all the subsequent values to be immensely large as well.

$$n = 2, \quad k = 2, \quad m = 4$$
$$w = 3, \quad h = 1, \quad l = 1$$
$$a = 17, \quad z = 3$$
$$x = 577, \quad y = 34$$

$$u = 84906869365454991485065820432991050648764346311890882321588090836248175717896238173603034916784966177173904562967610959612092029388630473803843852076197861167198886928455326699452583251545474926317027639388 17$$

$$v = 5003185258136129826178235128632666199004705973932442072989870120777917763935005658194508973623157529455523695790167416173392826240953424271748409267993883773240752959804131526189509041314193661115553507335 76$$

$$r = 4328014929183503309842763952104382525090576102017683454143486263648717788871112161067914337044253918214120844109141363471793102284561785702204506287191940980312070034432639728537637578991517007885426909 46$$

$$p = 7491782591069558072211690038205092704302736439284489616610713897316015504520256309435561904422202897985932755555965672906949296710761512394456810477311575985341078258393117061716404404548130140557384791710 721$$

$$q = 120$$

$$
\begin{aligned}
b = {} & 1018882432385459897820789845195892607785172155742690587859057090034978108614754 8 \\
& 580832364190014195941260868547556113315153451043526635656856461262249143743340 0 \\
& 638664314146392039343099901854569911580433167265805 7
\end{aligned}
$$

$$
\begin{aligned}
s = {} & 20762428220474225208176300961924058026222724135960311541444061750093998278585806629 \\
& 33762780688216827548878048136428860965347729327754116337869470710007218247767327 09 \\
& 35270432557349787021375442968571330496505558449491813660405577110658007240563041389 \\
& 00687547311720001151892650373384311316957397036679943773715593638455571124033660 64 \\
& 06344604022364181786907040098237513426405193321175660 \\
& 6452542846189888489360035517191294030497
\end{aligned}
$$

$$
\begin{aligned}
t = {} & 20377648647709197956415796903917852155703443114853811757181141800699562172295097161 \\
& 66472838002839188252173709511222663030690208705327131371292252449828748668012773286 \\
& 28292784078686199803709139823160866334531611 4
\end{aligned}
$$

$$
\begin{aligned}
c = {} & 24453178377251037547698956284701422586844131737824574108617370160839474606754116593 9 \\
& 97674056034070259026084514134671956368282504463925576455507029397944984016153279435 3 \\
& 95134089442343976445096778779303960143793776 0
\end{aligned}
$$

$$
\begin{aligned}
d = {} & 14983565182139116144423380076410185408605472878568979233221427794632031009040512618 8 \\
& 71123808844405795971865511111931345813898593421523024788913620954623151970682156516 78 \\
& 6234123432808809096260281114769583421443
\end{aligned}
$$

$$
e = 33, \quad f = 2, \quad g = 59
$$

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