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SOME PARTITION THEOREMS AND ULTRAFILTERS ON OMEGA.

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SOME PARTITION THEOREMS AND ULTRAFILTERS ON ω.

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ABSTRACT

Let η represent the order type of the rational numbers. An unpublished result of F. Galvin asserts the partition relation of $\eta \longrightarrow [\eta]^2_3$ where we use the standard partition arrow notation: given a 'coloring' $f: [Q]^2 \longrightarrow 3$ of pairs of rationals with three colors there exists a large subset $X \subseteq Q$ which avoids a color. The subset is large in that X has order type η, and X avoids a color in that there is $c \in 3$ such that $p \in [X]^2 \Longrightarrow f(p) \neq c$.

Proving this theorem leads naturally to the inductive construction of an order type η set where the steps of induction are labelled by the nodes of a full binary tree of height ω. A translation of the Galvin theorem entirely into the language of binary trees was stated by James Baumgartner. A definition of the 'big subsets' of an infinite full binary tree was given, corresponding to the order type η subsets of Q. In the language of trees, various generalizations or modifications of the original theorem are easily conjectured. For example, the same definition of 'big subset' can be applied to the ternary tree to give a partition theorem for pairs in which seven colors can be reduced to six on a big subset. By changing the definition of 'big subset', different partition theorems result, each theorem with its characteristic number of 'essential colors'. Partition theorems for triples, quadruples, etc. including polarized partition theorems are also naturally conjectured in the tree context. While some of the simpler cases of these various tree theorems can be proved 'directly' in analogy to

the proof of the Galvin theorem, the more complicated cases are most easily deduced from a version of the Halpern-Lauchli partition theorem as extended by K. Milliken in his unpublished paper A Ramsey Theorem for Trees.

Let $\mathcal{H} \subseteq \mathcal{P}(\omega)$ be a co-ideal, $\mathcal{H} = \mathcal{P}(\omega) - I$, where I is an ideal including the finite sets. If $\mathcal{H}$ satisfies a combinatorial property like $\mathcal{H} \to [\mathcal{H}]^2_6$, there seems hope for an ultrafilter, $\mathcal{U}$, such that $\mathcal{U} \to [\mathcal{U}]^2_6$. With CH, all possible partitions of $[\omega]^2$ can be enumerated in order type ω_1. The desired ultrafilter can be built using ω_1-induction provided the partial order $\langle \mathcal{H}, \subseteq \rangle / _I$ is countably complete [i.e., for any countable $(X \subseteq \mathcal{H})(\exists y \in \mathcal{H})$ such that $(\forall x \in X)(y - x \in I)$]. Although the simple tree theorems like Galvin's theorem do not yield directly a countably complete co-ideal, such co-ideals can be defined on a countable disjoint union of trees. In this way for example, a countably complete co-ideal $\mathcal{H}$ such that $\mathcal{H} \to [\mathcal{H}]^2_6$ is built and then a q-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longrightarrow [\mathcal{U}]^2_6$ and $\mathcal{U} \not\longrightarrow [\mathcal{U}]^2_5$.

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I would like to acknowledge a great debt to my thesis advisor, James Baumgartner, for his invaluable guidance. The idea of using Galvin's theorem, $\eta \rightarrow [\eta]^2_3$, as a tool for constructing a partition ultrafilter is his. Our constructions of a non-p-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \rightarrow [\mathcal{U}]^2_6$ (Theorem 7.22) and a p-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \rightarrow [\mathcal{U}]^2_4$ (Theorem 7.28) are specific reformulations of his results. These constructions served as a motivating force behind much of the thesis.

The proof of Theorem 7.6 using a theorem of Mathias is due to J. Baumgartner.

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INTRODUCTION

The combinatorial results of this thesis can be broadly classified as Ramsey type theorems or partition theorems. The most basic infinite and finite Ramsey theorems follow.

<u>The Infinite Ramsey Theorem</u> Given a function, $\kappa : [\omega]^2 \longrightarrow 2$ from the 2-element subsets of ω as domain, there exists an infinite subset $S \subseteq \omega$ such that κ is constant on $[S]^2$.

The function κ is referred to as a partition of $[\omega]^2$ into two colors and the set S is said to be homogeneous with respect to the partition κ.

<u>The Finite Ramsey Theorem</u> For any $h, n \in \omega$ there exists $r \in \omega$ such that for any partition $\kappa : [r]^n \longrightarrow 2$ of the n-element subsets of r there exist $H \subseteq r$ with $|H| = h$ and H is homogeneous with respect to κ (i.e., κ is constant on $[H]^n$).

In Chapter 1 we introduce a basic partition theorem for infinite trees due to Halpern and Lauchli [5] and prove some equivalences with similar theorems. In Chapter 2 we present and prove some consequences of the Halpern Lauchli Theorem due to Keith Milliken [7]. One of his theorems concerns partitions of finite subsets of an infinite tree and the infinite Ramsey theorem is a special case of this theorem.

In Chapter 3 the structural notions used by Milliken to define embeddings between trees are modified and some new partition theorems result.

In Chapter 4 one of the results of Chapter 3 is reformulated as the result $\eta \longrightarrow [\eta]^2_3$ due to F. Galvin and generalizations to Galvin's theorem follow from the reformulation.

Chapter 5 describes a standard procedure for constructing ultrafilters with specific combinatorial properties using the continuum hypothesis.

In Chapter 6 the Milliken partition result for partitions of the finite subsets of a tree is generalized to a partition result for the finite subsets of an infinite set of trees.

Chapter 7 uses the partition results of the previous chapter to construct square bracket partition ultrafilters with a great variety of properties. The techniques illustrated in this chapter go beyond the specific ultrafilter constructions which are presented as examples. More detailed information can easily be deduced about the ultrafilters which are constructed and further modifications to the partition theorems of Chapter 3 and Chapter 6 are easily conjectured.

CHAPTER 0

STANDARD NOTATION

Terminology and notation which is not specifically defined is quite standardized in modern set theory and logic texts (eg. [3] and [4]).

We will use the convention which identifies an ordinal with the set of its ordinal predecessors so in particular the symbol "0" will denote the least ordinal, the empty set, and the empty sequence. The letters n,m,r will invariably denote natural numbers and this will usually be indicated as $n,m,r \in \omega$.

A sequence will often be indicated with angle brackets, so for example $\langle 1,0,1 \rangle$ denotes the function $f \in {}^{3}2$ defined by $f(0) = 1$, $f(1) = 0$ and $f(2) = 1$. The concatenation notation $f^\frown 0$ denotes the extension of f to the function $g \in {}^{4}2$ defined by $f(i) = g(i)$ for $i \in 3$ and $g(3) = 0$. The restriction of g to the domain 3 is indicated by $g \restriction 3$ so in our example $g \restriction 3 = f$.

The restriction symbol is also used to restrict a model to a smaller similarity type. We will give specific definitions of another use of the restriction symbol in the text.

Given a function $f: A \longrightarrow B$ from the set A to the set B, typically an element of A will be denoted $a \in A$ with image $f(a) \in B$. We use $f''A$ to denote $\{f(a): a \in A\}$ usually but the notation $f(A) = \{f(a): a \in A\}$ is also used.

The cardinality of a set A is denoted $|A|$ and $\mathcal{P}(A)$ denotes the power set. $[A]^n$ means $\{B \subseteq A: |B| = n\}$. We use

$A - B$ to indicate $\{a \in A: a \notin B\}$. The symbol "$\exists!$" means "there exists a unique ...".

A tree is a partially ordered set, $\langle X; \prec \rangle$, such that X has a unique minimal element (called the root of X) and for any $x \in X$, $\{y \in X: y \prec x\}$ is well ordered. We will be interested only in finite or height ω trees so $(\forall x \in X)(|\{y \in X: y \prec x\}| \in \omega)$. For a height ω tree every node $x \in X$ has a well defined immediate predecessor except the root. The degree of a node in X is the number of its immediate successors in X. Typically we will be interested in infinite trees where all nodes have finite but non-zero degree. More of these standard tree definitions can be found in Drake [4] and Milliken [7] .

CHAPTER 1

A PARTITION THEOREM OF HALPERN AND LAUCHLI

We begin with a presentation of some known partition theorems on trees. As various relationships are pointed out between these theorems, a notation and vocabulary will evolve which will then be used to state and prove some new results. First we define a canonical set of objects on which partition theorems are considered.

Definitions 1.1 Let $\mathcal{K}$ be the collection of subsets $X \subseteq \bigcup_{n\epsilon\omega} {}^n\omega$ such that

(i) X is non-empty and closed under $\subseteq$-predecessors.

(ii) Every $f\epsilon X$ has a non-empty initial segment of immediate successors, $f^\frown 0$, $f^\frown 1$, $f^\frown 2$, ... $f^\frown n-1$

Any such $X \epsilon \mathcal{K}$ when structured by the inclusion relation, $\subseteq$, is a finitely branching, height ω tree, without maximal nodes. The tree partial ordering will be denoted $\preceq$ and $\prec$ will denote the strict partial order, $f \prec g \leftrightarrow f \subsetneq g$.

We define

$X(n) = \{f\epsilon X: |f| = n\}$ = the n'th level of X.

$X\restriction n = \{f\epsilon X: |f| < n\}$ = the first n-levels of X.

Note $X(0) = X\restriction 1 = \{0\}$ for all $X\epsilon\mathcal{K}$.

Theorems and definitions stated for canonical objects will have obvious restatements for objects isomorphic to canonical objects but it is often notationally and conceptually convenient to deal with definite concrete objects rather than axiomatically defined structures. Definitions of additional structure on the objects of $\mathcal{K}$ will be needed soon, and these definitions are particularly easy to write down and understand in the context of objects $X \subseteq \bigcup_{n\in\omega} {}^{n}\omega$.

The following definitions are based on concepts or similar definitions in the paper A Partition Theorem by Halpern and Lauchli [5].

Definition 1.2 Given $Y \in \mathcal{K}$ and $A, B \subseteq Y$, we say

A supports B iff $(\forall b\in B)(\exists a\in A)(a \preceq b)$

B dominates A iff $(\forall a\in A)(\exists b\in B)(a \preceq b)$

A is level in Y iff $(\exists n\in\omega)(A \subseteq Y(n))$.

Typically 'A is level in Y' is abbreviated by 'A is level' when the context Y is clear. If A is level (in Y) we say

B is a matrix over A (in Y) iff B dominates $\{y\in A\colon \exists a\in A\ (a \prec y\ \&\ |a| + 1 = |y|\}$ but no proper subset of B dominates this set.

B is a level matrix over A (in Y) iff B is both level (in Y) and B is a matrix over A (in Y).

B is a matrix (in Y) iff for some level matrix A over Y(0), B is a matrix over A.

B is a level matrix (in Y) iff B is level (in Y) and B is a matrix (in Y).

With these definitions we state a deceptively simple looking combinatorial theorem.

Theorem 1.3 Given $Y \in \mathcal{K}$ let $\mathcal{S} = \{A \subseteq Y : A$ is a matrix over $Y(o)\}$ For any partition $c : \mathcal{S} \to 2$ of $\mathcal{S}$ into 2 'colors', there exists a matrix in Y which is homogeneous with respect to c. That is to say, for some matrix B in Y, c is constant on $\mathcal{P}(B) \cap \mathcal{S}$.

A proof of Theorem 1.3 seems to require the statement of a much more complicated theorem as the basis for inductive constructions. The Theorem 1 in the Halpern Lauchli paper [5] can be regarded as an example of one such inductive hypothesis. Another complicated theorem suitable for inductive proof is the theorem of Richard Laver, "A pigeon hole principal for trees" which is stated and proved by Keith Milliken in his paper A Ramsey Theorem for Trees [7] . Following a somewhat circuitous route, we will state these two theorems and some generalizations. It will be trivially apparent that Theorem 1.3 follows from any of the more complicated theorems. Conversely, we will prove these theorems from Theorem 1.3 (while we never actually prove Theorem 1.3 here). The Halperin Lauchli proof of their Theorem 1 is very difficult. In comparison, all of our deductions from Theorem 1.3 to various theorems in chapter 1 are quite immediate. It is for this reason that we consider all of the results in this chapter as "closely related" or even "equivalent".

Our objective in Chapter 1 is to state these various known theorems and demonstrate their close relationships. Theorem 1.3 is now assumed as a basis for all of the following results.

Theorem 1.4 Given $Y \in \mathcal{K}$ and $\mathcal{S} = \{A \subseteq Y : A$ is a matrix over $Y(0)\}$ there exists $n \in \omega$ such that for every partition $\kappa : \mathcal{S} \to 2$, there is a homogeneous matrix B with respect to κ such that $B \subseteq Y \upharpoonright n$.

proof: Suppose the conclusion fails so we can define for each $n \in \omega$, the non-empty sets, C_n, of 'counter-example colorings', $C_n = \{\kappa : \mathcal{S} \longrightarrow 2$ such that for every matrix $B \subseteq Y \upharpoonright n$ κ is not constant on $\mathcal{P}(B) \cap \mathcal{S}\}$. If $n < m$ then $Y \upharpoonright n \subseteq Y \upharpoonright m$ so $C_n \supseteq C_m$. If we regard 2 as a discrete topological space, and ${}^{\mathcal{S}}2$ as the Tychonoff product topology, we have a chain $C_0 \supseteq C_1 \supseteq \ldots$ of non-empty closed subsets of a compact space. Let $\kappa \in \bigcap_{n\in\omega} C_n$. By Theorem 1.3 there is a homogeneous matrix, B, with respect to κ. Since B is a finite subset of Y, for some $n \in \omega$, $B \subseteq Y \upharpoonright n$ and hence $\kappa \notin C_n$. But this contradicts $c \in \bigcap_{n\in\omega} C_n$. □

Corollary 1.5 Given $Y \in \mathcal{K}$ and $\mathcal{S} = \{A \subseteq Y : A$ is a matrix over $Y(0)\}$ there exists $n \in \omega$ such that for every partition $\kappa : \mathcal{S} \to 2$ there is a level matrix $B \subseteq Y(n)$ which is homogeneous with respect to κ.

proof: Let n be as in Theorem 1.4 and let $\kappa : \mathcal{S} \longrightarrow 2$ be given. Let $\phi : Y \upharpoonright n \longrightarrow Y(n)$ be any fixed map such that $(\forall y \in Y \upharpoonright n)(y \prec \phi(y) \in Y(n))$. We extend the definition of ϕ to act on subsets of $Y \upharpoonright n$ in the usual way, so in particular for any A $\varepsilon\, \mathcal{S} \cap \mathcal{P}(Y{\upharpoonright}n)$ we have $\phi(A)\ \varepsilon\, \mathcal{S} \cap \mathcal{P}(Y(n))$. Define $\pi' : \mathcal{S} \cap \mathcal{P}(Y\ n) \longrightarrow 2$ for A $\varepsilon\, \mathcal{S} \cap \mathcal{P}(Y\ n)$ by $\pi'(A) = \pi(\phi(A))$. By Theorem 1.4 there is a matrix $B \subseteq Y{\upharpoonright}n$ which is homogeneous with respect to π'. Since $\phi(B) \subseteq Y(n)$ dominates B but no proper subset does, $\phi(B)$ is also a matrix in Y. By the definition of π' from π, in fact the level matrix $\phi(B) \subseteq Y(n)$ is homogeneous with respect to π. □

This finite consequence of Theorem 1.3 can now be turned back to give the stronger result:

<u>Theorem 1.6</u> Given $Y \varepsilon \mathcal{K}$ let $\mathcal{S} = \{A \subseteq Y :$ A is a matrix over $Y(0)\}$. For any partition $\pi : \mathcal{S} \longrightarrow 2$ there is a level matrix in Y which is homogeneous with respect to π.

<u>proof</u>: Clear.

Since the matrix asserted to exist here is level, the theorem could be stated in terms of a partition of $\mathcal{S}' = \{A \subseteq Y :$ A is a level matrix over $Y(0)\}$ without any real loss of power.

In order to strengthen this result further we define a k-matrix over A where A is level in Y. Recall that a matrix over A dominates all the immediate successors of elements a ε A.

<u>Definition 1.7</u> Given level $A \subseteq Y$ and $k \in \omega$, <u>B is a k-matrix over A</u> in Y iff B dominates $\{y \in Y : (\exists a \in A)(a \preccurlyeq y \text{ and } |a| + k = |y|\}$ but no proper subset of B does.

Hence a 1-matrix over A is just a matrix over A, and a 0-matrix over A is a minimal dominating set. Note that

$\mathcal{S} = \{A \subseteq Y : A \text{ is a 1-matrix over } Y(0)\}$

$= \{A \subseteq Y : A \text{ is a 0-matrix over } Y(1)\}$.

<u>Theorem 1.8</u> Given $Y \in \mathcal{K}$ let $\mathcal{S}' = \{A \subseteq Y : A \text{ is a level 1-matrix over } Y(0)\}$. For any partition $\pi : \mathcal{S}' \longrightarrow 2$, there exists some $A \in \mathcal{S}'$ such that $\forall k \in \omega$ there is a level k-matrix over A in Y which is homogeneous with respect to π.

<u>proof</u>: Suppose the conclusion fails, so to each $A \in \mathcal{S}'$ we can assign $\#(A)$ = the greatest $k \in \omega$ such that there is a homogeneous, level, k-matrix over A. We define a new tree tree Y' (not a canonical tree, $Y \notin \mathcal{K}$) with each level $Y'(n)$ of Y' chosen as some level $Y(f(n))$ of Y. The function $f: \omega \to \omega$ is defined inductively by

$$f(0) = 0$$

$$f(n+1) = 1 + f(n) + \sup\{\#(A) : A \in \mathcal{S}' \cap \mathcal{P}(Y(f(n)))\}$$

For example, $f(1) = 1$ and $f(2) = 1 + 1 + \#(Y(1) - Y(0))$, and since $\mathcal{S}' \cap \mathcal{P}(Y(f(n)))$ is finite, $f(n+1) \in \omega$ and f is an increasing function.

The $\prec$-structure on $Y' = \bigcup_{n\in\omega} Y(f(n))$ is inherited from $\langle Y; \prec^{Y} \rangle$, and clearly $\langle Y'; \prec^{Y'} \rangle$ is a finitely branching height ω tree without terminal nodes. Since $Y'(1) = Y(1)$ and $Y'(n)$ is level in Y,

$\mathcal{S}' \cap \mathcal{P}(Y') = \{A \subseteq Y' : A$ is a level 1-matrix over $Y'(0)\}$. κ restricts to a partition on $\mathcal{S}' \cap \mathcal{P}(Y')$ and Theorem 1.6 gives a level homogeneous matrix B in Y' (technically we must stretch the implications of Theorem 1.6 here since Y' is not canonical). Hence for some $n \in \omega$ there exists $A \in \mathcal{S}' \cap \mathcal{P}(Y\ (n)) = \mathcal{S}' \cap \mathcal{P}(Y(f(n))$ such that B is a level 1-matrix over A in Y', and κ is constant on $\mathcal{P}(B) \cap \mathcal{S}'$. Put $\ell = f(n+1) - f(n)$. By the construction of Y', B is a homogeneous, level, ℓ-matrix over A in Y. But $\ell > \sup\{\#(A) : A \in \mathcal{S}' \cap \mathcal{P}(Y(f(n))\}$ so $\ell > \#A =$ the greatest $k \in \omega$ such that there is a homogeneous, level, k-matrix over A, and we have a contradiction. $\square$

At this point we begin to develop a very general framework for stating partition results. The point of view reflected by our choice of notation is inspired by the paper of Nesetril and Rodl, Partitions of Finite Relational and Set Systems [8]. After setting this notation we will return in this chapter to statements of known results using the notation.

The set of objects $\mathcal{K}$ will now become the set of objects of a category which we also denote by $\mathcal{K}$. In order

To define the morphisms of the category, the objects of $\mathcal{K}$ are formalized as first order structures in which a fixed similarity type, σ , is interpreted. A category, $\mathcal{F}$, of finite objects will also be needed. The objects of $\mathcal{K}$ and $\mathcal{F}$ are all substructures of a single structure, T, which we now define.

<u>Definition 1.9</u> Let $T = \bigcup_{n\in\omega} {}^{n}$. The binary function symbols

Δ 'pass-meet'

$\wedge$ 'meet'

and the binary relation symbols

$\prec$ 'tree partial order'

$<$ 'levelwise left-right linear order'

$\ll$ 'level partial order'

$\prec_n$ $(n\in\omega)$ 'n-extension partial order'

comprise the similarity type, σ.

The interpretation of these symbols in T is defined for $x, y \in T$ by

$$x \wedge^T y = \bigcup\{z \in T: z \subseteq x \ \& \ z \subseteq y\}$$

$$x \Delta^T y = \begin{cases} x \restriction |y| & \text{if } |x| > |y| \\ y \restriction |x| & \text{if } |y| > |x| \\ x \wedge^T y & \text{if } |x| = |y| \end{cases}$$

$x \prec^T y$ iff $x \subseteq y$

'y properly extends x'

$x <^T y$ iff $|x| < |y|$ or

$(|x| = |y| \,\&\, x \neq y \,\&\, x(|x \wedge^T y|) < y(|x \wedge^T y|))$

'x is younger than y'

$x <<^T y$ iff $|x| < |y|$

'x is lower than y'

$x \prec_n^T y$ iff $\widehat{xn} \subseteq y$

'y is an n-extension of x'

Let $\langle T;\sigma\rangle$ be an abbreviated notation for the fixed structure $\langle T; \prec^T, <^T, <<^T, \prec_n^T, \Delta^T, \wedge^T\rangle_{n\in\omega}$.

Definition 1.10 Let the objects of $\mathcal{K}$ be the set of all substructures of $\langle T;\sigma\rangle$ with universe $X \subseteq T$ satisfying

(i) X is non-empty and closed under $\prec$-predecessors in T.

(ii) Every $x \in X$ has a non-empty initial segment of immediate $\prec$-successors; $\widehat{x0}, \widehat{x1}, \widehat{x2}, \ldots \widehat{xn-1}$. We will call this $n \in \omega - \{0\}$ the degree of x in X, which is denoted $\deg_X(x) = |\{y \in X: x \prec y \,\&\, |x| + 1 = y\}|$.

Let the objects of $\mathcal{F}$ be the set of all substructures of $\langle T;\sigma\rangle$ with universe $A \subseteq T$ satisfying

(i) A is non-empty and closed under $\prec$-predecessors in T

(iii) A is finite.

Let $\mathcal{C} = \mathcal{F} \cup \mathcal{K}$

Comments 1.11 We will use the symbol X (or A etc.) ambiguously to denote both the structure

$\langle X; \prec^X, <^X, <<^X, \prec_n^X, \Delta^X, \wedge^X\rangle_{n\in\omega}$ and its universe.

Typically the superscripts on the symbols $\prec$, $<$, $\ll$ etc. will be suppressed whenever no confusion results.

Condition (i) guarantees the closure of a set under the two operations Δ^T and Λ^T, so we have sensible definitions of substructures of T.

Condition (ii) insures that objects in $\mathcal{K}$ are finitely branching, height ω, trees without maximal nodes. The initial segment condition on immediate successors implies that for any $n \in \omega$, an object $X \in \mathcal{K}$ satisfies $(\forall x \in X)(\exists y \in X\ (x \prec_{n+1} y) \rightarrow \exists z \in X\ (x \prec_n z))$. This property is not necessarily satisfied by objects in $\mathcal{F}$.

Condition (iii) insures that any object $A \in \mathcal{F}$ is a finite substructure of some $X \in \mathcal{K}$, but of course not all finite substructures of an $X \in \mathcal{K}$ are objects in $\mathcal{F}$ since they may not satisfy (i). For any $n \in \omega$ and $X \in \mathcal{K}$, $X \restriction n \in \mathcal{F}$ but not all finite objects are realized in this fashion since $A \in \mathcal{F}$ may have $a, b \in A$ with $a \prec_1 b$ without having any $c \in A$ such that $a \prec_0 c$.

To complete the definitions of $\mathcal{F}$, $\mathcal{K}$, and $\mathcal{C}$ as categories, the morphisms must be specified.

Definition 1.12 Let $A \in \mathcal{F}$, $X \in \mathcal{K}$ and $Z \in \mathcal{C} = \mathcal{F} \cup \mathcal{K}$. We define the morphisms from A to Z in the category $\mathcal{C}$ by

$$\mathcal{C}(A,Z) = \{ \langle A, \phi, Z \rangle : \phi : A \hookrightarrow Z \text{ is an isomorphic embedding of the } \sigma\text{-structure } A \text{ into the } \sigma\text{-structure } Z \}.$$

When the domain is an infinite object there is an additional constraint on the morphism. $\mathcal{C}(X,Z) = \{ \langle X, \phi, Z\rangle \mid \phi : X \hookrightarrow Z$ is an isomorphic embedding of the σ-structure X into the σ-structure Z and $(\forall x \varepsilon X)(\deg_X(x) = \deg_Z(\phi(x))\}$.
Composition of morphisms is defined as composition of maps, and the identity maps are the identity morphisms, so $\mathcal{C}$ is a category.

By requiring that $\mathfrak{F}$ and $\mathcal{K}$ are full subcategories of $\mathcal{C}$ (i.e. for $X,Y \varepsilon \mathcal{K}$ and $A,B \varepsilon \mathfrak{F}$ $\mathcal{K}(X,Y) = \mathcal{C}(X,Y)$ and $\mathfrak{F}(A,B) = \mathcal{C}(A,B)$) we have defined the morphisms of the categories $\mathcal{K}$ and $\mathfrak{F}$ as well.

<u>Comments 1.13</u> The words <u>isomorphic embedding</u> are being used here in the sense of model theory [3], so for example, given $\langle A, \phi, Z\rangle \varepsilon \mathcal{C}(A,Z)$, and $a,b,c \varepsilon A$ and $n \varepsilon \omega$ then

$$a \prec_n^A b \iff \phi(a) \prec_n^Z \phi(b)$$

$$a \wedge^A b = c \iff \phi(a) \wedge^Z \phi(b) = \phi(c) \text{ etc.}$$

where $\phi : A \hookrightarrow Z$ is a one-one map. In this case we say that ϕ preserves the diagram of A (or ϕ preserves the similarity type) as contrasted with a homomorphism $\psi : A \to Z$ which only preserves the positive diagram (or weakly preserves the similarity type) e.g. $a \prec_n^A b \Longrightarrow \psi(a) \prec_n^Z \psi(b)$ but not conversely.

By enlarging the similarity type interpreted by objects in $\mathcal{C}$, the morphisms which we have defined in cases depending on whether the domain of the morphism is finite, could have been defined uniformly as monomorphisms between the structures. The only advantage would be a more uniform appearance of our definitions of morphism at the expense of a more complicated structure on the objects.

Our choice of similarity type of objects in $\mathcal{C}$ was designed to make available some useful notation in addition to providing the structure which constrains the notion of a morphism between objects. Thus economy has not been attempted -- e.g. is definable from Δ. On the other hand, the similarity type is not exhaustive and we may refer to defined relations -- e.g. "x is on the same level as y" $x \not< y$ & $y \not< x$. A minimal similarity type from which the entire structure on objects $Z \varepsilon \mathcal{C}$ is definable, can be extracted in various ways. When later we discuss isomorphisms to non-canonical structures it will be useful to give an example of such a minimal set.

For any $Z \varepsilon \mathcal{C}$, $<^Z$ is a well ordering of Z (of finite type if $Z \varepsilon \mathfrak{F}$, and of type ω if $Z \varepsilon \mathcal{K}$) any any morphism strongly preserves < so there are no non-trivial automorphisms in the category $\mathcal{C}$. In fact,

<u>Lemma 1.14</u> Every isomorphism $\phi: Z \hookrightarrow\!\!\!\rightarrow W$ between objects $Z,W \varepsilon \mathcal{C}$ is an identity map.

proof: Suppose ϕ is not an identity map, and look at the $<$-least $z \in Z$ such that $\phi(z) \neq z$. Let $w \in Z$ be the immediate $\prec$-predecessor of z, so $\phi(w) = w$. Condition (i) in the definition of the objects in $\mathfrak{F}$ and $\mathcal{K}$ along with the fact that ϕ preserves $\prec_n$ for all $n \in \omega$ forces $\phi(z) = z$, a contradiction. $\square$

This lemma justifies our use of the word 'canonical' in referring to the objects of $\mathcal{C}$.

Definition 1.15 For $Z \in \mathcal{C}$ define
height $(Z) = \sup \{|z| + 1 : z \in Z\}$.

Note for $X \in \mathcal{K}$, height $(X) = \omega$ and for any $n \in \omega$ height $(X \upharpoonright n) = n$. If $\phi: Z \to W$ is a morphism between objects $Z, W \in \mathcal{C}$, since ϕ preserves $<<$, each level, $Z(n)$, of Z is mapped into some level, $W(m)$, of W. Thus ϕ naturally induces a non-decreasing map,
$\overline{\Phi}$: height $(Z) \hookrightarrow$ height (W), which satisfies
$\Phi(Z(n)) \subseteq Y(\overline{\Phi}(n))$.

The preservation (by a morphism) of the binary function parameters $\wedge$ and Δ, follows from the preservation of the relations $\prec_n$ $(n \in \omega)$, and $<<$. The function parameters are included in the similarity type for notational convenience and when the notion of sub-object is later defined they will impose a closure constraint.

Loosely we think of a morphism of $\mathcal{K}$ as a map which preserves the tree partial order $\prec$, takes levels to levels, and preserves degree. This is Millken's definition of a "strong embedding" [7]. Here we have the further condition that ϕ preserves the 'levelwise left-to-right linear order', $<$.

Now that we have a well defined class of canonical objects $\mathcal{C} = \mathcal{F} \cup \mathcal{K}$, it is easy to define the general class of objects in which we will be interested.

<u>Definition 1.16</u> Let $\overline{\mathcal{K}}$ be the category of objects X such that for some object $Y \in \mathcal{K}$, X is isomorphic to Y (isomorphic as σ-structures).

Let $\overline{\mathcal{F}}$ be the category of objects A such that for some object $B \in \mathcal{F}$, A is isomorphic to B.

Let $\overline{\mathcal{C}} = \overline{\mathcal{F}} \cup \overline{\mathcal{K}}$.

Note that since the only isomorphisms of $\mathcal{C}$ are the identity maps, any object $Z \in \overline{\mathcal{C}}$ corresponds by a unique <u>canonical isomorphism</u> $\rho_Z: Z \leftrightarrow\!\!\!\rightarrow W$, to a unique canonical object $W \in \mathcal{C}$. The morphisms of the category $\overline{\mathcal{C}}$ are thus defined naturally via this correspondence.

Notation and terminology which was defined for objects in $\mathcal{C}$ will be translated by the unique canonical isomorphism to objects in $\overline{\mathcal{C}}$.

Example 1.17 Let $X = \{f \in T : |f|$ is even and $(\forall i \in \mathrm{dom}(f)) \left[(i \text{ is even} \rightarrow f(i) \in 2) \;\&\; (i \text{ is odd} \rightarrow f(i) = 0)\right]\}$ with structure inherited from $\langle T;\sigma\rangle$. X is isomorphic to $\bigcup_{n\in\omega} {}^{n}z$ (structured by inheritance from $\langle T;\sigma\rangle$). Let $Y = \bigcup_{n\in\omega} {}^{n}z$. The canonical isomorphism $\rho_X: X \hookrightarrow\!\!\!\rightarrow Y$ is defined by $\rho_X(f) = g$ where $(\forall n\in\omega)(g(n) = f(2n))$. Following this isomorphism we have for instance $X\restriction 3 = \{f \in X: |f| \leq 4\}$. The natural inclusion map $i: X \hookrightarrow Y$, is a degree preserving embedding so $\langle X, i, Y\rangle \in \overline{C}(X,Y)$.

Definition 1.18 Given $W,Z \in \overline{C}$ such that $W \subseteq Z$, if the natural inclusion map $i : W \hookrightarrow Z$ (defined by $(\forall w\in W)(i(w) = w)$) is a morphism, $\langle W,i,Z\rangle \in \overline{C}(W,Z)$, then we say *W is a subobject of Z*, which is denoted $W \subset\!\subset Z$.

Even though "W is a substructure of Z" may hold (i.e. the natural inclusion is an isomorphic embedding of σ-structures), it does not necessarily follow that the inclusion map is a morphism -- the condition of degree preservation must also be met when $W \in \overline{\mathcal{K}}$. For finite $W \in \overline{\mathcal{F}}$, the notions of sub-object and sub-structure do coincide, and the role of the operation parameters Δ and ∧ is thus clarified.

<u>Lemma 1.19</u> Given $Z \in \overline{\mathcal{C}}$, the finite subset $A \subseteq Z$ is a sub-object, $A \in \overline{\mathcal{F}}$ & $A \subset\subset Z$, iff A is a substructure of Z.

<u>proof</u>: Clear.

<u>Lemma 1.20</u> Given $Z \in \overline{\mathcal{C}}$, the infinite subset $Y \subseteq Z$ is a sub-object, $Y \in \overline{\mathcal{K}}$ & $Y \subset\subset Z$ iff Y is a substructure of Z and $(\forall y \in Y)(deg_Y(y) = deg_Z(y))$.

<u>proof</u>: Clear.

Given $Y \in \overline{\mathcal{K}}$, it is easy to see how the corresponding canonical object $Z \in \mathcal{K}$, and the isomorphism $\rho_Y : Y \longleftrightarrow Z$ can be determined from the limited structure $\langle Y; \prec, < \rangle$; the map $\rho_Y : Y \longleftrightarrow T$ is simply defined by $<$-induction to preserve $\prec$ and $<$ and then Z is taken as the image. As a consequence of this construction, it is clear that for any $Y \in \overline{\mathcal{K}}$, the entire structure $\langle Y; \sigma \rangle$ is definable from $\langle Y; \prec, < \rangle$ (but obviously not definable from $\langle Y; \prec \rangle$ or $\langle Y; < \rangle$). It is useful to know some necessary and sufficient conditions on the structure $\langle Y; \prec, < \rangle$ so that it extends to an object $\langle Y; \sigma \rangle \in \overline{\mathcal{K}}$. The following technical lemma gives one such set of conditions and the very obvious but tedious proof is included to illustrate some of the technical properties satisfied by structures of $\overline{\mathcal{K}}$.

Lemma 1.21 The following are equivalent.

(I) There exists a unique isomorphic ebbedding $\rho_Y: \langle Y; \prec, < \rangle \hookrightarrow \langle T; \prec, < \rangle$ whose image when structured by inheritance from $\langle T;\sigma \rangle$ is an object of $\mathcal{K}$.

(II) $\langle Y; \prec, < \rangle$ extends uniquely to $\langle Y;\sigma \rangle \; \varepsilon \; \overline{\mathcal{K}}$.

(III) Define for x,y ε Y;

$x{<<}y \leftrightarrow |\{z \; \varepsilon \; Y: z \prec x\}| < |\{z \; \varepsilon \; Y: z \prec y\}|$.

$\langle Y: \prec, <, << \rangle$ satisfies

(a) $\langle Y;\prec \rangle$ is a height ω, finitely branching tree without maximal nodes.

(b) $\langle Y;< \rangle$ is a linear order (not reflexive)

(c) $(\forall x,y \; \varepsilon \; Y) \; (x{<<}y \rightarrow x<y)$

(d) $(\forall x,y \; \varepsilon \; Y) \; (\forall x',y' \; \varepsilon \; Y) \; (x \not\prec y \; \& \; y \not\prec x \; \& \; x' \not\prec y' \; \& \; y' \not\prec x' \; \& \; x' \prec x \; \& \; y' \prec y \; \& \; x' < y' \rightarrow x<y)$.

proof: I⇒II clear. II⇒III clear.

III⇒I: The map $\rho_Y: Y \hookrightarrow T$ is constructed inductively using the conditions a-thru-d. The $\prec$-minimal element of Y (which exists by a) is first mapped to 0 ε T, and this map (with singleton domain) is called ρ_0. Having defined inductively the finite maps $\rho_0 \quad \rho_1 \quad \ldots \quad \rho_n$, conditions (a) and (b) show that $\{y \; \varepsilon \; Y : (\exists y' \; \varepsilon \; \mathrm{dom}(\rho_n)) \; (y$ is an immediate $\prec$-successor of $y'\}$ is a finite set linearly ordered by <. Let y_{n+1} be the <-least element of this set, where y_{n+1} is the immediate successor of say y'.

Clearly y_{n+1} must be mapped by ρ_{n+1} to the $<$-least element, t_{n+1}, of $\{t \in T : \rho_n(y') \prec t\}$ in order to insure that the image, $(\bigcup_{n\in\omega} \rho_n)(Y)$, will be canonical. Hence we must put $\rho_{n+1} = \rho_n \cup (y_{n+1}, t_{n+1})$ so that $\rho_Y = \bigcup_{n\in\omega} \rho_n$ has a canonical image and satisfies

(i) $(\forall y', y \in Y)\ (y' \prec^Y y \leftrightarrow \rho_Y(y') \prec^T \rho_Y(y))$

(ii) $(\forall x,y,z \in Y)$

$(x,y$ are immediate $\prec$-successors of $z \rightarrow (x <^Y y \leftrightarrow \rho_Y(x) <^T \rho_Y(y))$

(iii) $(\forall x,y \in Y)\ (x \ll^Y y \leftrightarrow \rho_Y(x) \ll^T \rho_Y(y))$

The relation $<$ on a canonical object can be thought of as a piecing together of the linear orders on the (finite) immediate successor sets, to form the total $<$-ordering of type ω. So far we know that $\rho_Y: Y \rightarrow T$ preserves the relation $<^Y$ restricted to any set of immediate successors of a single node. The purpose of conditions (IIIc) and (IIId) is to show that $<^Y$ restricted to immediate successor sets is pieced together to form the linear order $<^Y$ in exactly the same way this piecing together is done in a canonical object. A pair of distinct nodes x,y falls into one of three cases:

<u>Case 1</u> x,y are immediate successors (in Y) of some $z \in Y$. We know already $x <^Y y \leftrightarrow \rho_Y(x) <^T \rho_Y(y)$.

<u>Case 2</u> $x \ll^Y y$ (or $y \ll^Y x$). From (c) $x \lessgtr^Y y$ and $x \ll^Y y \longrightarrow \rho_Y(x) \ll^T \rho_Y(y) \rightarrow \rho_Y(x) \lessgtr^T \rho_Y(y)$.

i.e. $x \lessgtr^Y y \rightarrow \rho_Y(x) \lessgtr^T \rho_Y(y)$.

<u>Case 3</u> $x \not\lessdot^Y y$ & $y \not\lessdot^Y x$ & $\exists x', y', z \in Y$ such that x' and y' are immediate successors (in Y) of z and $x' \prec^Y x$ and $y' \prec^Y y$. Suppose without loss of generality that $x <^Y y$ and hence using (d) $x' \not\lessdot^Y y'$. From Case 1, $\rho_Y(x') <^T \rho_Y(y')$ so $\rho(x) <^T \rho(y)$ i.e., $x <^Y y \rightarrow$ [illegible] $\rho_Y(x) <^T \rho_Y(y)$. Since $<^Y$ and $<^T$ are strict linear orders we conclude $(\forall x,y \in Y)\ (x <^Y y \leftrightarrow \rho_Y(x) <^T \rho_Y(y))$ and $\rho_Y: Y \hookrightarrow T$ is an isomorphic embedding of the structure $\langle Y; \prec, <\rangle$ with $\rho_Y{}''Y \in \mathcal{K}$. □

<u>Corollary 1.22</u> Suppose $X \in \overline{\mathcal{K}}$, a function $f: \omega \rightarrow \omega$ and sets $Y(n) \subseteq X(f(n))$ satisfy $|Y(0)| = 1$ and $(\forall n \in \omega)\ (\forall y \in Y(n))\ (\exists z \in Y(n+1))\ (y \prec^X z)$. Then the set $Y = \bigcup_{n\in\omega} Y(n)$ with structure $\langle Y; \prec, <\rangle$ inherited from $\langle X; \prec^X, <^X\rangle$ extends uniquely to an object $\langle Y; \sigma\rangle \in \overline{\mathcal{K}}$.

Proof. With $\lessdot^Y$ defined as usual from $\prec^Y$, note that $(\forall y,z \in Y)\ (y \lessdot^Y z \leftrightarrow y \lessdot^X z)$. It is easily checked that conditions (III a-d) are satisfied by $\langle Y; \prec, <\rangle$. □

Note in the corollary we are <u>not</u> claiming $Y \subset\subset X$. Although $\langle Y; \prec, <\rangle$ is a substructure of $\langle X; \prec, <\rangle$, in general the natural inclusion will fail to preserve degrees and fail to preserve the meet operation.

Before developing our notation further, we can give a preliminary statement of a theorem which is credited to R. Laver and D. Pincus (by Milliken [7]).

<u>Theorem 1.23</u> Given $Y \in \mathcal{R}$, let

$\mathcal{S}' = \{A \subseteq Y : A$ is a level 1-matrix over $Y(0)\}$.

For any partition $\pi : \mathcal{S}' \to 2$, there exists $X \in \overline{\mathcal{R}}$ such that $X \subset\subset Y$, $X(0) = Y(0)$ and π is constant on $\mathcal{P}(X) \cap \mathcal{S}'$.

<u>proof</u>: By Theorem 1.8, there exists $A \in \mathcal{S}'$ such that $\forall k \in \omega$ there is a level homogeneous k-matrix over A in Y. For $k' > k$, any k'-matrix over A includes a k-matrix over A so we can assume that for a fixed color, $c \in 2$, and for every $k \in \omega$, we have chosen B_k -- a level homogeneous k-matrix over A in the color c (i.e. $\pi''(\mathcal{P}(B_k) \cap \mathcal{S}') = \{c\}$ for all B_k). Let $a, b_k \in \omega$ satisfy $A \subseteq Y(a)$ and $B_k \subseteq Y(b_k)$. Define $f : \omega \to \omega$ by induction:

$f(0) = 0$

$f(n+1) = b_{f(n)} + 1 - a$.

$X \subseteq Y$ is defined a level at a time by induction

$X(0) = Y(0)$

$X(n+1) = \{b \in B_{f(n)} : \exists y \in X(n) \quad y \prec b\}$

$X = \bigcup_{n \in \omega} X(n)$

Now for every $y \in Y(b_{f(n)}+1)$ $(\exists! \ b \in B_{f(n+1)})$ $(y \preccurlyeq b)$ since $B_{f(n+1)}$ is a $b_{f(n)}+1-a$ matrix over A.

Hence for any $x \in X(n+1) \subseteq B_{f(n)} \subseteq Y(b_{f(n)})$, $\deg_X(x) = \deg_Y(x)$ and X with structure inherited from Y is a sub-object, $X \in \bar{\mathcal{K}}$, $X \subset\subset Y$, $X(0) = Y(0)$. By construction $c''(\mathcal{P}(X) \cap \mathcal{S}') = c \in 2$. □

<u>Corollary 1.24</u> Replace $c: \mathcal{S}' \longrightarrow 2$ in the theorem by $c: \mathcal{S}' \longrightarrow r$, a partition into finitely many colors $r \in \omega$.

<u>proof</u>: The corollary results from grouping the colors together with a definition of $\bar{c} : \mathcal{S}' \longrightarrow 2$ like

$$\bar{c}(B) = \begin{cases} 0 \text{ if } c(B) = 0 \\ 1 \text{ otherwise} \end{cases}$$

where $B \in \mathcal{S}'$. The theorem yields $X_0 \in \bar{\mathcal{K}}$ such that $X_0 \subset\subset Y$, $X_0(0) = Y(0)$ and $|\bar{c}''(\mathcal{S}' \cap \mathcal{P}(X_0))| = 1$ so $|c''(\mathcal{S}' \cap \mathcal{P}(X_0))| < r$. By iterating finitely many times we get $X_{r-1} \in \bar{\mathcal{K}}$ such that $X_{r-1} \subset\subset X_{r-2} \subset\subset \ldots X_0 \subset\subset Y$ where $X_{r-1}(0) = Y(0)$ and c is constant on $\mathcal{S}' \cap \mathcal{P}(X_{r-1})$. □

<u>Definitions and Notation 1.25</u>

The notation used by Nešetřil and Rödl in [8] for stating partition theorems places the emphasis on the morphisms of a category. In a fixed category $\mathcal{D}$ of objects and morphisms they define for $f, f' \in \mathcal{D}(X,Y)$ (= the morphisms from the object X to the object Y) $f \sim f'$ <u>iff</u> $\exists$ an automorphism $h \in \mathcal{D}(X,X)$ such that $f = f' \circ h$. The equivalence class of f with respect to ~ is denoted $[f]_X$ and $\{[f]_X : f \in \mathfrak{D}(X,Y)\}$ is denoted $\binom{Y}{X}_{\mathfrak{D}}$ or $\binom{Y}{X}$

A partition theorem typically involves a partition of some $\binom{Y}{X}$. In our category $\bar{\mathfrak{C}}$, for any $X \in \bar{\mathfrak{C}}$ $\bar{\mathfrak{C}}(X,X) = \{id_X\}$. Hence the equivalence classes under ~ in $\mathfrak{C}$ are singletons. This allows $\binom{Y}{X}$ to be interpreted simply as "the set of sub-objects of Y isomorphic to X" (since a morphism $f \in \bar{\mathfrak{C}}(X,Y)$ can be identified by its domain and image in this category).

This dual interpretation of $\binom{Y}{X}$ will be exploited, and sometimes elements of $\binom{Y}{X}$ will be denoted as sub-objects of Y while at other times these elements will be denoted as morphisms (though technically an element of $\binom{Y}{X}$ is an equivalence class of morphisms.) The notation $\binom{Y}{X}$ generalizes the notation for the binomial coefficeent $\binom{n}{m}$ = the number of ways to choose copies of m (i.e., m-element subsets) in n.

The $\binom{Y}{X}$ notation is made more powerful by borrowing the notion of a 'map of pairs' or a 'map of triples' etc. from topology. In a category $\mathfrak{D}$ where there is a well defined notion of sub-object (i.e. the concept of a canonical inclusion morphism) we first define for $X' \subset\subset X$ and $Y' \subset\subset Y$ (where $X', X, Y', Y \in \mathfrak{D}$)
$\mathfrak{D}((X,X'), (Y,Y')) = \{f \in \mathfrak{D}(X,Y) : f$ is an extension of some $f' \in \mathfrak{D}(X',Y')\}$.
Here extension means $f \circ i_{X',X} = i_{Y',Y} \circ f'$ where

$i_{X',X} \in \mathcal{D}(X',X)$ and $i_{Y',Y} \in \mathcal{D}(Y',Y)$ are the canonical inclusion morphisms.

This gives the morphisms for a category of pairs of objects (X,X') where $X' \subset\subset X$. We let
$\binom{Y,Y}{X,X} = \{[f]_{(X,X')} : f \in \mathcal{D}((X,X'),(Y,Y'))\}$ where the equivalence class $[f]_{(X,X')}$ is with respect to automorphisms in $\mathcal{D}((X,X'),(X,X'))$. Perhaps a more natural notation would be

$\binom{(Y,Y')}{(X,X')}$ but the inner parenthesis can be eliminated without confusion.

For triples of objects (X,X',X'') and (Y,Y',Y'') where $X' \subset\subset X$, $X'' \subset\subset X$, $Y' \subset\subset Y$, and $Y'' \subset\subset Y$, the generalization of the above notation to
$\begin{pmatrix} Y, & Y', & Y'' \\ X, & X', & X'' \end{pmatrix}$ is obvious.

In a category like $\bar{\mathcal{C}}$ where morphisms from a given object correspond to sub-objects of the target, this notation is particularly transparent. For example, given $Y \in \bar{\mathcal{C}}$, $\begin{pmatrix} Y & , & Y\rceil 1 \\ Y\rceil 2, & & Y\rceil 1 \end{pmatrix}$ is essentially the set of sub-objects of Y isomorphic to $Y\rceil 2$, such that <u>the</u> isomorphism carries $Y\rceil 1$ into $Y\rceil 1$. That is to say, the sub-objects of Y isomorphic to $Y\rceil 2$ which include Y(0). An element of $\begin{pmatrix} Y & , & Y\rceil 1 \\ Y\rceil 2, & & Y\rceil 1 \end{pmatrix}$ will sometimes be denoted as an object $A \in \bar{\mathfrak{F}}$ (or even as a pair $(A,Y(0))$ and sometimes as a morphism, f.

Note that $A \in \begin{pmatrix} Y & , Y\restriction 1 \\ Y\restriction 2, & Y\restriction 1 \end{pmatrix}$ iff $A - Y(0)$ is a level 1-matrix over $Y(0)$. With this observation, the Laver-Pincus theorem (1.24) can be restated.

<u>Theorem 1.26</u> For any $Y \in \bar{\mathcal{K}}$ and any finite partition $\mathcal{C}: \begin{pmatrix} Y & , Y\restriction 1 \\ Y\, 2, & Y\restriction 1 \end{pmatrix} \longrightarrow r$ there exists $X \in \bar{\mathcal{K}}$ such that $X \subset\subset Y$, $X\restriction 1 = Y\restriction 1$ and $\mathcal{C}$ is constant on $\begin{pmatrix} X & , X\restriction 1 \\ Y\restriction 2, & X\restriction 1 \end{pmatrix}$.

By an obvious extension of our sub-object notation to the situation of pairs etc., the conclusion of this theorem can be stated,
"there exists $X \in \bar{\mathcal{K}}$ such that $(X, X\restriction 1) \subset\subset (Y, Y\restriction 1)$ and X is homogeneous for $\mathcal{C}$."

<u>Corollary 1.27</u> Given any $Y \in \bar{\mathcal{K}}$, $n \in \omega$ and a finite partition $\mathcal{C}: \begin{pmatrix} Y & , Y\restriction n \\ Y\restriction n+1, & Y\restriction n \end{pmatrix} \longrightarrow r$ there exists $X \in \bar{\mathcal{K}}$ such that $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$ and X is homogeneous for $\mathcal{C}$.

<u>proof</u>: Assume $n > 0$. The idea is simply to consolidate $Y\restriction n$ to a single point which acts as the root of a new object $Y' \in \bar{\mathcal{K}}$, in which the sub-objects being colored are $\begin{pmatrix} Y' & , Y'\restriction 1 \\ Y'\restriction 2, & Y'\restriction 1 \end{pmatrix}$. The details of the obvious construction of Y' are justified by Corollary 10.8.

In case $n = 0$, the object $Y' \in \overline{\mathcal{K}}$ is formed by adjoining a new root, Λ, to Y -- i.e., $\Lambda \prec y$ for all $y \in Y$. Clearly $\begin{pmatrix} Y & , Y\upharpoonright 0 \\ Y\upharpoonright 1, & Y\upharpoonright 0 \end{pmatrix} = \begin{pmatrix} Y \\ Y\upharpoonright 1 \end{pmatrix}$ corresponds to $\begin{pmatrix} Y' & , Y'\upharpoonright 1 \\ Y'\upharpoonright 2, & Y'\upharpoonright 1 \end{pmatrix}$ and the corollary follows from this correspondence. $\square$

An asymetric version of Theorem 1.26 follows.

<u>Theorem 1.28</u> Given $Y \in \overline{\mathcal{K}}$ and a finite partition $c : \begin{pmatrix} Y & , Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix} \longrightarrow r$, if $c(Y\upharpoonright 2) = 0$, then there exists $X \in \overline{\mathcal{K}}$ such that $(X, X\upharpoonright 1) \subset\subset (Y, Y\upharpoonright 1)$ and either

(i) $X\upharpoonright 2 = Y\upharpoonright 2$ with $c''\begin{pmatrix} X & , X\upharpoonright 1 \\ Y\ 2, & Y\upharpoonright 1 \end{pmatrix} = \{0\}$ or

(ii) $c''\begin{pmatrix} X & , X\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix} = \{i\}$ where $i \in r - \{0\}$.

<u>proof</u>: Assume $r = 2$. Let

$$\begin{pmatrix} Y\upharpoonright 3, & Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix} - \{Y\upharpoonright 2\} = \{ A_0, A_1, \ldots, A_{m-1} \}.$$

Define $\bar{c} : \begin{pmatrix} Y & , Y\upharpoonright 2 \\ Y\upharpoonright 3, & Y\upharpoonright 2 \end{pmatrix} \to {}^m 2$ for $B \in \begin{pmatrix} Y & , Y\upharpoonright 2 \\ Y\upharpoonright 3, & Y\upharpoonright 2 \end{pmatrix}$ by defining the i'th coordinate of $\bar{c}(B)$ (where $i \in m$) as $\bar{c}(B)(i) = c(A)$ where $A \in \begin{pmatrix} B & , Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix}$ is the unique member which dominates A_i. By Corollary 1.27, there exists $Z \in \overline{\mathcal{K}}$ such that $(Z, Z\upharpoonright 2) \subset\subset (Y, Y\upharpoonright 2)$ and Z is homogeneous for $\bar{c}$. In case the homogeneous color of Z with respect to $\bar{c}$ has a one in any coordinate, say the i'th coordinate is one, then define $X \in \bar{\mathcal{K}}$ by

$X = \{z \in Z : z = Y(0) \text{ or } (\exists a \in A_i - Y(0))(a \preceq z)\}$.

Since X is closed under Δ, and

$(\forall x \in X)(\deg_X(x) = \deg_Z(x) = \deg_Y(x))$, we have $X \in \bar{\mathcal{K}}$ and $(X, X\upharpoonright 1) \subset\subset (Y, Y\upharpoonright 1)$. It is easily checked that

$$c''\begin{pmatrix} X & , \upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix} = \{1\}.$$

In case the homogeneous color of Z with respect to $\bar{c}$ is zero in all coordinates, then put $X = Z$. We have $(X, X\upharpoonright 2) \subset\subset (Y, Y\upharpoonright 2)$ and for any $A \in \begin{pmatrix} X & , X\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix}$

either $A = Y\upharpoonright 2$ so $c(A) = 0$ by assumption, or there exists $B \in \begin{pmatrix} X & , X\upharpoonright 2 \\ Y\upharpoonright 3, & Y\upharpoonright 2 \end{pmatrix}$ such that $A \in \begin{pmatrix} B & , Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix}$ and for some $i \in m$, A must dominate A_i so $0 = \bar{c}(B)(i) = c(A)$.

The strengthening of this result to the case $r > 2$ follows by applying the $r = 2$ case to the partition $c' : \begin{pmatrix} Y & , Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix} \longrightarrow 2$ defined for $A \in \begin{pmatrix} Y & , Y\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1 \end{pmatrix}$

by $c'(A) = \begin{cases} 0 \text{ if } c(A) = 0 \\ 1 \text{ otherwise} \end{cases}$

If $(Z, Z\upharpoonright 2) \subset\subset (Y, Y\ 2)$ is found as in conclusion (i) of the theorem (with color 0 with respect to c') then put $X = Z$ and we are done. If $(Z, Z\upharpoonright 1) \subset\subset (Y, Y\upharpoonright 1)$ is found as in conclusion (ii) of the theorem (with color 1 with respect to c') then one further application of Theorem 1.26 is required to reduce $(Z, Z\upharpoonright 1)$ to say $(X, X\upharpoonright 1) \subset\subset (Z, Z\upharpoonright 1)$

which is homogeneous with respect to κ and hence has color $\kappa''\begin{pmatrix} X & , X\upharpoonright 1 \\ Y\upharpoonright 2, & Y\upharpoonright 1\end{pmatrix} = \{i\}$ where $i \in r - \{0\}$. $\square$

As in the case of the Laver-Pincus result, all of the previous theorems can be upgraded to an asymetric many color version. One such theorem is worth mentioning.

<u>Theorem 10.14</u> Given $Y \in \mathcal{K}$ and a finite partition $\kappa : \mathcal{S}' \longrightarrow r$ where $\mathcal{S}' = \{A \subseteq Y :$ A is a level 1-matrix over $Y(0)\}$, then either

(i) $(\forall k \in \omega)$ there is a level homogeneous k-matrix, B_k, over $Y\upharpoonright 2$ with color 0 (i.e. $\kappa''(\mathcal{P}(B_k) \cap \mathcal{S}') = \{0\}$)

or

(ii) there exists $A \in \mathcal{S}'$ such that $(\forall k \in \omega)$ there is a level homogeneous k-matrix, B_k, over A in color $i \in r - \{0\}$ (i.e., $\kappa''(\mathcal{P}(B_k) \cap \mathcal{S}') = \{i\}$).

<u>proof</u>: Theorem 1.28 immediately gives an asymmetric version of Theorem 1.6 (which deals only with the existence of a level matrix in Y). The asymmetric version of 1.6 gives our Theorem 1.29 by the same argument which generalized 1.6 to 1.8 (which deals with the existence of level k-matrices over some $A \in \mathcal{S}'$). $\square$

This theorem is a slight strengthening of the original Halpern - Lauchli Theorem 1 in [5]. Their theorem dealt with only 2 colors and produced k-matrices rather than level k-matrices (so their coloring was of $\mathcal{S} = \{A \subseteq Y : A \text{ is a 1-matrix over } Y\upharpoonright 1\}$). Also in their statement of conclusion (ii), rather than claiming the existence of a single $A \in \mathcal{S}'$ which the k-matrices dominate $(k \in \omega)$, they only claim the existence of some $h \in \omega$ such that $(\forall k \in \omega)(\exists A \in \mathcal{S}' \cap \wp(Y(h))$ and there exists a homogeneous k-matrix over A with color 1).

The only aspect of Theorem 1.29 which does not follow is a completely trivial way from the original Halpern and Lauchli version is the change from non-level to level matrices. This change is accomplished by the compactness argument of Theorem 1.4 (essentially the Halpern -Lauchli Theorem 2) and then the strengthening from matrix to k-matrix takes place in Theorem 1.8.

The analogue in the new context (of level k-matrices) to the Halpern and Lauchli Corollary 1 (which strengthens their Theorem 1) is easily obtained.

<u>Corollary 1.30</u> Given $Y \in \mathcal{K}$ and a finite partition $c : \mathcal{S}' \longrightarrow r$ as in Theorem 1.29, suppose also that $W \subseteq Y$ satisfies

$$\forall n \exists m (\forall y \in Y(n)) (\exists w \in W(m)) (y \prec w).$$

Then either

(i) $\forall k \in \omega$ there is a level homogeneous k-matrix, B_k, over $Y\upharpoonright 2$ with color 0 and $B_k \subseteq W$ or

(ii) there exists $A \in \mathcal{S}'$ such that $(\forall k \in \omega)$ there is a level homogeneous k-matrix, B_k, over A in color $i \in r - \{0\}$ and $B_k \subseteq W$.

<u>proof</u>: The analogous Corollary 1 of Halpern and Lauchli is proved by what they call 'the principle', and the idea here is to modify their 'principle' so it respects levels.

Let $\phi : Y \longrightarrow W$ satisfy $(\forall y \in Y)(y \prec \phi(y))$ and $(\forall x,y \in Y)((x << y) \leftrightarrow \phi(x) << \phi(y))$. The existence of such a map ϕ follows by a simple inductive construction from the assumptions about W. Define $\bar{\kappa} : \mathcal{S}' \longrightarrow r$ for $A \in \mathcal{S}'$ by $\bar{\kappa}(A) = \kappa(\phi''A)$ where we note that the conditions on ϕ guarantee $\phi''A \in \mathcal{S}'$. Apply Theorem 1.29 to $\bar{\kappa} : \mathcal{S}' \longrightarrow r$ so k-matrices $\bar{B}_k$, $k \in \omega$ exist either as in case (i) or case (ii) of Theorem 1.29. Let $B_k = \phi''(\bar{B}_k)$ so clearly $B_k \subseteq W$ and B_k is a level k-matrix. For any $A \in \mathcal{P}(B_k) \cap \mathcal{S}'$, let $\bar{A} = \phi^{-1}(A)$ so $\bar{A} \in \mathcal{P}(\bar{B}_k) \cap \mathcal{S}'$ and $\kappa(A) = \kappa(\phi''\bar{A}) = \bar{\kappa}(\bar{A})$. Hence the B_k $(k \in \omega)$, satisfy either condition (i) or (ii) of the corollary. □

Now from Theorem 1.29 (or its non-level version, Halpern and Lauchli Theorem 1) our starting point, Theorem 1.3, follows trivially and we have a circle of 'equivalent' theorems. To establish the truth of all these theorems we refer the reader to Halpern Lauchli [5] or to Milliken [7] (where Milliken proves the Laver Pincus result, Theorem 1.3, from the Halpern - Lauchli machinery).

CHAPTER 2

A FORMALIZATION OF SOME TREE PARTITION RESULTS OF MILLIKEN

All of the theorems of Chapter 1 essentially dealt only with the tree partial ordering $\prec$, and the level partial ordering $\ll$ (which can be defined from $\prec$). A justification for the elaborate structure is now finally seen in the statement of the following theorem due to Keith Milliken (which appears in [7] as Theorem 4.3).

Theorem (Milliken) For any $Y \in \overline{\mathcal{K}}$ and any $A \in \overline{\mathcal{F}}$ and any finite partition $\kappa : \binom{Y}{A} \to r$, there exists $X \in \overline{\mathcal{K}}$ such that $X \subseteq\subseteq Y$ and κ is constant on $\binom{X}{A}$.

Remark 2.1 We can assume in the hypothesis of the theorem that Y and A are canonical objects without any loss, and this statement may clarify the structural considerations which define the set of sub-objects being partitioned.

Example 2.2 Consider a partition of pairs of nodes from an infinite binary tree. Let $Y = \bigcup_{n \in \omega} {}^{n}2 \subseteq \langle T;\sigma \rangle$ inherit structure from $\langle T;\sigma \rangle$, and let $\kappa : [Y]^2 \to r$ be a finite partition of the unordered pairs from Y. In general, a pair of nodes is not a substructure of Y since it may not be closed under pass-meet (the only 2-element substructures of Y, are the substructures isomorphic to either $\{0, \langle 0 \rangle\} \in \mathcal{F}$ or to $\{0, \langle 1 \rangle\} \in \mathcal{F}$).
By considering the closure under pass-meet, a pair can be

classified into one of the following 7 isomorphism types:

$/ = \{0, \quad 0 \quad \}$ $\qquad\qquad$ $\backslash = \{0, \quad 1 \quad \}$

$$\Lambda = \text{closure}\ \{\langle 0\rangle , \langle 1\rangle \}$$
$$= \{\langle 0\rangle , \langle 1\rangle , 0\}$$

 $= \text{closure}\ \{\langle 0,0\rangle , \langle 1\rangle \}$
$\{ \langle 0,0\rangle , \langle 1\rangle , \langle 0\rangle , 0\}$

 $= \text{closure}\ \{\langle 0\rangle , \langle 1,1\rangle \}$
$= \{\langle 0\rangle , \langle 1,1\rangle , \langle 1\rangle , 0\}$

 $= \text{closure}\ \{\langle 0,1\rangle , \langle 1\rangle \}$
$\{\langle 0,1\rangle , \langle 1\rangle , \langle 0\rangle , 0 \}$

 $= \text{closure}\ \{\langle 0\rangle , \langle 1,0\rangle \}$
$= \{\langle 0\rangle , \langle 1,0\rangle , \langle 1\rangle , 0\}$

The Milliken theorem says that any finite parititon of each of the given sets $\binom{Y}{/}$, $\binom{Y}{\Lambda}$ etc. can be reduced by some $X \in \overline{\mathcal{K}}$, $X \subset\subset Y$. The given partition $\pi : [Y]^2 \to r$ naturally induces partitions on $\binom{Y}{/}$, $\binom{Y}{\Lambda}$ etc. and by iterating the theorem 7 times, $\pi : [Y]^2 \to r$ can be reduced by some $X \in \overline{\mathcal{K}}$, $X \subset\subset Y$ so that $|\pi''[X]^2| \leq 7$. Furthermore, since any $X \subset\subset Y$, $X \in \overline{\mathcal{K}}$ is isomorphic to Y, all seven types of substructure generated by closing a pair under pass-meet are <u>essential</u> -- for any one of these 7 finite structures $A \in \{/, \backslash, \Lambda, \text{etc.}\}$, and any $X \subset\subset Y$ where $X \in \overline{\mathcal{K}}$, we have $\binom{X}{A} \neq 0$.

<u>Remark 2.3</u> Similar applications of the theorem to ternary trees (etc.) and partitions of n-tuples are easily imagined. It is useful to consider the most trivial such application -- the case of an "infinite unary tree". Let $Y = \bigcup_{n\in\omega} {}^n1 \subseteq \langle T;\sigma\rangle$, and consider a finite partition of n-tuples $\mathcal{C} : \binom{Y}{Y\upharpoonright n} \to r$. The existence of $X \in \overline{\mathcal{K}}$ such that $X \subset\subset Y$ and X is homogeneous for $\mathcal{C}$ is just the familiar Ramsey theorem for a partition of n-tuples from an infinite set.

In order to prove Millikens theorem, as usual in partition theory it is necessary to state and prove a more elaborate theorem which is more appropriate for carrying out inductive arguments. The exact form of the elaboration is not very critical and perhaps not even very interesting in itself. For reasons of economy therefore, we proceed with little further motivation to state and prove a string of lemmas which are pieced together to give a general theorem (Theorem 2.9) from which Milliken's theorem follows as a special case. It is useful to consider the meaning of each lemma when applied to the unary tree and a partition of n-tuples. The lemmas and proofs become trivial in this case, thus highlighting the relationships between the lemmas so that an analogy to a simple proof of Ramsey's theorem is seen.

<u>Lemma 2.4</u> Given $X,Y \in \overline{\mathcal{K}}$ and $n \in \omega$, if $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$ then $X\restriction n+1$ is isomorphic to $Y\restriction n+1$.

<u>proof 2.4</u> This proposition follows essentially from the degree preservation condition which the sub-object relation, $X \subset\subset Y$, entails for $X,Y \in \overline{\mathcal{K}}$. $\square$

<u>Lemma 2.5</u> Given $Y \in \overline{\mathcal{K}}$ and $A \subset\subset Y$ where $A \in \overline{\overline{\mathcal{F}}}$, let $A' = A\restriction \text{height}(A)-1$, and suppose n satisfies $A' = A \cap Y\restriction n \neq A \cap Y\restriction n-1$. For any finite partition $c : \begin{pmatrix} Y, & A' \\ A, & A' \end{pmatrix} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}$ such that $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$ and X is homogeneous for c.

<u>proof</u>: This is an easy consequence of Corollary 1.27. We need to induce a partition $\overline{c} : \begin{pmatrix} Y & , Y\restriction n \\ Y\restriction n+1, & Y\restriction n \end{pmatrix} \longrightarrow r$ in such a way that any $X \in \overline{\mathcal{K}}$ such that $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$ which is homogeneous for $\overline{c}$ is then also homogeneous for c.

Without loss of generality we can assume $(A, A') \subset\subset (Y\restriction n+1, A')$. Given $\overline{C} \in \begin{pmatrix} Y & , Y\restriction n \\ Y\restriction n+1, & Y\restriction n \end{pmatrix}$,

let $C = \{c \in \overline{C} : (\exists a' \in A'^{\sim} \cap Y(n-1))\ (\exists a \in A \cap Y(n))$
$a' \prec a \ \&\ a \preccurlyeq c\} \cup A'$.

Since $(\forall y \in Y(n))\ (\exists !\, c \in C)\ (y \preccurlyeq c)$, it is easily seen that $C \in \begin{pmatrix} \overline{C}, & A' \\ A, & A' \end{pmatrix}$. Define $\overline{c}(\overline{C}) = c(C)$, and use Lemma 2.4 to find $X \in \overline{\mathcal{K}}$ such that $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$

and X is homogeneous for $\bar{\mathcal{C}}$.

Given any $C \in \begin{pmatrix} X, & A' \\ A, & A' \end{pmatrix}$, clearly C extends to some $\bar{C} \in \begin{pmatrix} X, & X\upharpoonright n \\ X\upharpoonright n+1, & X\upharpoonright n \end{pmatrix}$ (typically there are many such extensions $\bar{C} \supseteq C$). But $\mathcal{C}(C) = \bar{\mathcal{C}}(\bar{C})$ so X is homogeneous for $\mathcal{C}$ also. $\square$

We need to consider partitions of a set more general than a set of all sub-objects isomorphic to a fixed object, so the following definitions are made.

<u>Definition 2.6</u> Given $\mathcal{S} \subseteq \overline{\mathcal{F}}$ and $Y \in \overline{\mathcal{K}}$, let $\binom{Y}{\mathcal{S}} = \{A \in \mathcal{S} \,.\, A \subset\subset Y\}$. Given a partition $\mathcal{C} : \binom{Y}{\mathcal{S}} \longrightarrow r$ we say <u>X reduces $\mathcal{C}$</u> when X satisfies $X \in \overline{\mathcal{K}}$, $X \subset\subset Y$, and $\mathcal{C}$ is constant on $\binom{X}{\mathcal{S}}$.

<u>$(X, X\upharpoonright n)$ reduces $\mathcal{C}$</u> means X reduces $\mathcal{C}$ and $(X, X\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$.

We say <u>X weakly reduces $\mathcal{C}$</u> iff X satisfies $X \in \overline{\mathcal{K}}$, $X \subset\subset Y$ and for any $A, B \in \binom{X}{\mathcal{S}}$

$(A\upharpoonright \text{height}(A)-1 = B\upharpoonright \text{height}(B)-1 \rightarrow c(A) = c(B))$.

In this context, for $A \in \binom{Y}{\mathcal{S}}$ we let A' denote $A\upharpoonright \text{height}(A)-1$ and $\mathcal{S}' = \{A' : A \in \mathcal{S}\}$.

<u>Lemma 2.7</u> Let $\mathcal{S} \subseteq \overline{\mathcal{F}}$, $Y \in \overline{\mathcal{K}}$ and $n \in \omega$ satisfy

(i) $\forall A \in \binom{Y}{\mathcal{S}}$ $(A' = A \cap Y\upharpoonright n \neq A \cap Y\upharpoonright n-1)$

(ii) $\forall A, B \in \binom{Y}{\mathcal{S}}$ $(A' = B' \rightarrow A$ is isomorphic to $B)$

For any finite partition $\mathcal{C} : \binom{Y}{\mathcal{S}} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}$ such that $(X, X\upharpoonright n)$ weakly reduces $\mathcal{C}$.

<u>proof</u>: The set $\{A' : A \in \binom{Y}{\mathcal{S}}\}$ is finite since $A' = A \cap Y \upharpoonright n$ and we denote this set $\{A_1', A_2', \ldots, A_k'\}$ where $k \in \omega$ and $A_1, A_2, \ldots, A_k \in \binom{Y}{\mathcal{S}}$ have been chosen such that $A_1 \upharpoonright \text{height}(A_1)-1 = A_1'$, $A_2 \upharpoonright \text{height}(A_2)-1 = A_2'$, etc. Conditions (i) and (ii) imply

$$\binom{Y}{\mathcal{S}} = \binom{Y\ ,\ A_1'}{A_1,\ A_1'} \cup \binom{Y\ ,\ A_2'}{A_2,\ A_2'} \cup \ \cdots \ \cup \binom{Y\ ,\ A_k'}{A_k,\ A_k'} .$$

Using (i) and Lemma 2.5, we have $X_1 \in \overline{\mathcal{K}}$ such that $(X_1, X_1 \upharpoonright n) \subset\subset (Y, Y \upharpoonright n)$ and $\mathcal{C}$ is constant on $\binom{X_1,\ A_1'}{A_1,\ A_1'}$.

Then $X_1 \upharpoonright n = Y \upharpoonright n$ and $A_2 = A_2 \cap X_1 \upharpoonright n \neq A_2 \cap X_1 \upharpoonright n-1$ so again using Lemma 2.5 we have $X_2 \in \overline{\mathcal{K}}$ such that $(X_2, X_2 \upharpoonright n) \subset\subset (X_1, X_1 \upharpoonright n)$ and $\mathcal{C}$ is constant on $\binom{X_2,\ A_2'}{A_2,\ A_2'}$ & $\binom{X_2,\ A_1'}{A_1,\ A_1'}$. Proceeding through the finitely many $A_1, A_2, \ldots, A_k$ we finally get $X_k \in \overline{\mathcal{K}}$ such that $(X_k, X_k \upharpoonright n) \subset\subset (X_{k-1}, X_{k-1} \upharpoonright n) \ldots \subset\subset (Y, Y \upharpoonright n)$ and $\mathcal{C}$ is constant on the sets $\binom{X_k,\ A_k'}{A_k,\ A_k'}$, $\binom{X_k,\ A_{k-1}'}{A_{k-1},\ A_{k-1}'}$, ..., $\binom{X_k,\ A_1'}{A_1,\ A_1'}$. Thus X_k weakly reduces $\mathcal{C}$. □

<u>Lemma 2.8</u> Let $\mathcal{S} \subseteq \overline{\mathcal{F}}$, $Y \in \overline{\mathcal{K}}$ and $n \in \omega$ satisfy

(i) $\forall A \in \binom{Y}{\mathcal{S}}$ $(A' \supsetneqq A \cap Y \upharpoonright n-1)$

(ii) $\forall A, B \in \binom{Y}{\mathcal{S}}$ $(A' = B' \rightarrow A$ is isomorphic to $B)$

For any finite partition $\mathcal{c} : \binom{Y}{\mathcal{S}} \to r$ there exists $X \in \overline{\mathcal{K}}$ such that $(X, X\upharpoonright n)$ weakly reduces $\mathcal{c}$.

<u>proof</u>: Put $X_n = Y$ and let

$\mathcal{S}_{n+1} = \{A \in \binom{X_n}{\mathcal{S}} : A' = A \cap X_n\upharpoonright n \neq A \cap X_n\upharpoonright n-1\}$.

The assumption (ii) and Lemma 2.7 gives $X_{n+1} \in \overline{\mathcal{K}}$ such that $(X_{n+1}, X_{n+1}\upharpoonright n)$ weakly reduces $\mathcal{c} : \binom{X_n}{\mathcal{S}_{n+1}} \to r$. Continue by induction to define

$\mathcal{S}_{k+1} = \{A \in \binom{X_k}{\mathcal{S}} : A' = A \cap X_k\upharpoonright k \neq A \cap X_k\upharpoonright k-1\}$

and let X_{k+1} satisfy $(X_{k+1}, X_{k+1}\upharpoonright k)$ weakly reduces $\mathcal{c} : \binom{X_k}{\mathcal{S}_{k+1}} \to r$. Put $X = \bigcap_{k \geq n} X_k$. Note

$X = \bigcup_{k \geq n} X_k\upharpoonright k$, $X \in \overline{\mathcal{K}}$, $X\upharpoonright k = X_k\upharpoonright k$, and $(X, X\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$.

To show that $(X, X\upharpoonright n)$ weakly reduces $\mathcal{c} : \binom{Y}{\mathcal{S}} \to r$, for any $A, B \in \binom{X}{\mathcal{S}}$ such that $A' = B'$ we must have $\mathcal{c}(A) = \mathcal{c}(B)$. Using assumption (i), for some $k \geq n$

$A' = A \cap X\upharpoonright k = A \cap X_k\upharpoonright k \neq A \cap X_k\upharpoonright k-1 = A \cap X\upharpoonright k-1$.

Hence $A \in \mathcal{S}_{k+1}$ and likewise $B \in \mathcal{S}_{k+1}$. But $(X_{k+1}, X_{k+1}\upharpoonright k)$ weakly reduces $\mathcal{c} : \binom{X_k}{\mathcal{S}_{k+1}} \to r$ and $(X, X\upharpoonright k) \subset\subset (X_{k+1}, X_{k+1}\upharpoonright k)$ so $A, B \in \binom{X_{k+1}}{\mathcal{S}_{k+1}}$ with $A' = B'$ and hence $\mathcal{c}(A) = \mathcal{c}(B)$. $\square$

<u>Theorem 2.9</u> Let $\mathcal{S} \subseteq \overline{\mathcal{F}}$, $Y \in \overline{\mathcal{K}}$, $n \in \omega$, $D \in \overline{\mathcal{F}}$ satisfy

(i) $D \subset\subset Y \upharpoonright n$, $D \not\subseteq Y \upharpoonright n-1$

(ii) $\forall A \in \binom{Y}{\mathcal{S}}$ $(A \cap Y \upharpoonright n = D)$

(iii) $\forall A,B \in \binom{Y}{\mathcal{S}}$ $(\text{height}(A) = \text{height}(B))$

(iv) $(\forall m \in \omega)$ $(\forall A,B \in \binom{Y}{\mathcal{S}}$ $)$

$(A \upharpoonright m = B \upharpoonright m \rightarrow A \upharpoonright m+1$ is isomorphic to $B \upharpoonright m+1)$.

For any finite partition $c : \binom{Y}{\mathcal{S}} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}$ such that $(X, X \upharpoonright n)$ reduces c.

<u>proof</u>: Using condition (iii), we let $h = \text{height}(A)$ where A is any member of $\binom{Y}{\mathcal{S}}$. Let $d = \text{height}(D)$. The proof is by induction on h-d.

<u>case 0</u> h-d = 0. Using (ii) and (iii), h-d = 0 implies $\binom{Y}{\mathcal{S}} = \{D\}$, and the assertion of the theorem is trivial.

<u>case k+1</u> h-d = k+1 and by induction we assume the theorem is true for $h-d \leq k$. Our conditions (i), (ii), (iv) and $h > d$ imply the condition (i) and (ii) of Lemma 2.8. Hence there exists $Y' \in \overline{\mathcal{K}}$ such that $(Y', Y' \upharpoonright n)$ weakly reduces $c : \binom{Y}{\mathcal{S}} \longrightarrow r$. Let $\mathcal{S}' = \{A' : A \in \mathcal{S}\}$. A partition $c' : \binom{Y}{\mathcal{S}'} \longrightarrow r$ is naturally induced from $c : \binom{Y}{\mathcal{S}} \longrightarrow r$ by defining $c'(A') = c(B)$ where $A' \in \binom{Y'}{\mathcal{S}'}$ and $B \in \binom{Y'}{\mathcal{S}}$ is any object such that $B \upharpoonright \text{height}(B)-1 = A'$. Now $\mathcal{S}' \subseteq \overline{\mathcal{F}}$, $Y' \in \overline{\mathcal{K}}$, $n \in \omega$, $D \in \overline{\mathcal{F}}$ satisfy (i) through (iv) where $\forall A' \in \binom{Y'}{\mathcal{S}'}$ $(\text{height}(A') = h-1 = h')$.
Since $h' - d = k$, by inductive hypothesis there exists $X \in \overline{\mathcal{K}}$ such that $(X, X \upharpoonright n)$ reduces $c' : \binom{Y'}{\mathcal{S}'} \longrightarrow r$. But for any $A,B \in \binom{X}{\mathcal{S}}$, we have $A',B' \in \binom{X}{\mathcal{S}'}$ and

$\mathcal{c}(A) = \mathcal{c}'(A') = \mathcal{c}'(B') = \mathcal{c}(B)$, so $(X, X\upharpoonright n)$ reduces $\mathcal{c} : \binom{Y}{\mathcal{J}} \longrightarrow r$. □

The main theorem of Milliken in [7], now follows easily as an application Theorem 2.9.

Theorem 2.10 (Milliken 4.3 in [7])

Given $Y \in \overline{\mathcal{K}}$, $A \in \overline{\mathcal{F}}$ and any finite partition $\mathcal{c} : \binom{Y}{A} \longrightarrow r$, there exists $X \in \overline{\mathcal{K}}$ such that X reduces $\mathcal{c}$.

proof: Form $\overline{Y} \in \overline{\mathcal{K}}$ by adjoining a new root, Λ, to Y -- i.e., we assume $\Lambda \notin Y$ and define for every $y \in Y$ $(\Lambda \prec_b y)$ so that $Y \cup \{\Lambda\}$ can be completed to an object $\overline{Y} \in \overline{\mathcal{K}}$ in the obvious (and unique) way. In the same way, given $B \in \binom{Y}{A}$ the structure $\overline{B} = B \cup \{\Lambda\}$ is defined by the requirement $(\forall b \in B)(\Lambda \prec_o b)$. Let $D = \{\Lambda\} \in \overline{\mathcal{F}}$, $n = 1$ and $\mathcal{J} = \{\overline{B} : B \in \binom{Y}{A}\}$. Now $\mathcal{J} \subseteq \overline{\mathcal{F}}$, $\overline{Y} \in \overline{\mathcal{K}}$, $n \in \omega$, and $D \in \overline{\mathcal{F}}$ satisfy conditions (i)-(iv) of theorem 2.9. Define $\overline{c}(\overline{B}) = c(B)$ where $B \in \binom{Y}{A}$, and we have $\overline{X} \in \overline{\mathcal{K}}$ such that $(X, X\upharpoonright 1)$ reduces $\overline{\mathcal{c}}$. Put $X = \overline{X} - \{\Lambda\}$ (with inherited structure) so clearly $X \in \overline{\mathcal{K}}$ and X reduces c. □

<u>Theorem 2.11</u> (Milliken 3.1 in [7])

Given $Y \in \overline{\mathcal{K}}$ and finite $k > 1$, let $\mathcal{S} = \{X{\restriction}k : (X,X{\restriction}1) \subset\subset (Y,Y{\restriction}1)$ where $X \in \overline{\mathcal{K}}\}$.

For any finite partition $\pi : \binom{Y}{\mathcal{S}} \to r$, there exists $X \in \overline{\mathcal{K}}$ such that $(X,X{\restriction}1)$ reduces π.

<u>proof</u>: Let $D = Y{\restriction}1$ and $n = 1$. It is easily checked that $\mathcal{S}$, Y, n, D satisfy conditions (i)-(iii) of Theorem 2.9. Condition (iv) follows from Lemma 2.4. Hence Theorem 2.9 gives the required $X \in \overline{\mathcal{K}}$ such that $(X,X{\restriction}1)$ reduces π. □

In addition to Milliken's theorems, other special applications of Theorem 2.9 can be found just by giving specific examples of $\mathcal{S} \subseteq \overline{\mathcal{F}}$, $Y \in \overline{\mathcal{K}}$, $n \in \omega$ and $D \in \overline{\mathcal{F}}$ which satisfy the hypothesis. In this sense, Theorem 2.9 is more general than 2.10 or 2.11 but the difference between our proof and Milliken's proof of 2.11 is essentially a matter of organization and emphasis rather than a difference in method or content. The objective here has been to write the proof in such a manner that its generalization (in the context of a more general category) will be easy to follow.

CHAPTER 3

THREE VARIATIONS ON MILLIKEN'S THEME

Using Theorem 2.10 as a basic combinatorial result concerning $\overline{\mathcal{K}}$ and $\overline{\mathcal{F}}$, similar results can be obtained by modifying the objects and/or morphisms under consideration. The morphisms of $\overline{\mathcal{K}}$ and $\overline{\mathcal{F}}$ respect levels, and as a first modification, we will drop this requirement.

<u>Definition 3.1</u> Let σ_1 be the similarity type $\sigma_1 = \{\prec, <, \prec_n, \wedge\}_{n\varepsilon\omega}$. Let $\overline{\mathcal{K}}_1 = \{X\upharpoonright\sigma_1 : X \varepsilon \mathcal{K}\}$ where $X\upharpoonright\sigma_1$ is the reduct of $X \varepsilon \mathcal{K}$ to a structure interpreting the similarity type σ_1 rather than σ.

<u>Definition 3.2</u> Let the objects of $\mathcal{F}_1$ be all substructures, A, of $\langle T; \prec, <, \prec_n, \wedge\rangle_{n\varepsilon\omega} = \langle T;\sigma_1\rangle$ such that

(i) $(\forall n\varepsilon\omega)\ (A \cap T(n) \neq 0 \longrightarrow (\forall m \leq n)\ |A \cap T(m)| = 1)$

(ii) $(\forall a \varepsilon A)(\forall t \varepsilon T - A)\ (t \prec a \longrightarrow t \prec_0 a)$

(iii) A is non-empty and finite.

Put $\mathcal{C}_1 = \mathcal{F}_1 \cup \mathcal{K}_1$

The morphisms for $A \varepsilon \mathcal{F}_1$, $X \varepsilon \mathcal{K}_1$, $Z \varepsilon \mathcal{C}_1$ are defined by

$\mathcal{C}_1(A,Z) = \{\langle A,\phi,Z\rangle : \phi$ is an isomorphic embedding of the σ_1-structure A into the σ_1-structure Z$\}$

$\mathcal{C}_1(X,Z) = \{\langle X,\phi,Z\rangle : \phi$ is an isomorphic embedding of the σ_1-structure X into the σ_1-structure Z and $(\forall x \varepsilon X)\ (\deg_X(x) = \deg_Z(x))\}$.

$\mathcal{F}_1$ and $\mathcal{K}_1$ receive their morphisms as full subcategories of $\mathcal{C}_1$. Composition of morphisms is defined of course as composition of maps.

To insure that we have well defined substructures of $\langle T;\sigma_1\rangle$ it must be checked that an object $Z \varepsilon \mathcal{C}_1$ is closed under $\wedge$ in T. This is clear for $X \varepsilon \mathcal{K}_1$. For $A \varepsilon \mathcal{F}_1$, suppose $x,y \varepsilon A$ but $x \wedge y \not\varepsilon T - A$. Then $x \wedge y \prec x$ & $x \wedge y \prec y$, so using (ii) we conclude $x \wedge y \prec_0 x$ & $x \wedge y \prec_0 y$, a contradiction.

So that we may treat $\mathcal{C}_1 = \mathcal{F}_1 \cup \mathcal{K}_1$ as canonical objects, we must verify that there are no non-trivial isomorphisms between objects.

<u>Lemma 3.3</u> Every isomorphism between objects of $\mathcal{C}_1$ is an identity map.

<u>proof</u>: Given infinite objects $W,Z \varepsilon \mathcal{K}_1$, let $\overline{W}, \overline{Z} \varepsilon \mathcal{K}$ be the unique expansions of W, Z respectively to objects of $\overline{\mathcal{K}}$ (using Lemma 1.21). Note in fact $\overline{W}, \overline{Z} \varepsilon \mathcal{K}$. Any isomorphism $f: W \longrightarrow Z$ is also an isomorphism $f: \overline{W} \longrightarrow \overline{Z}$ and hence by Lemma 1.14, f is the identity map.

There are no non-trivial automorphisms of an object $A \varepsilon \mathcal{F}_1$ (since $<^A$ is a well ordering). Suppose distinct A and B are members of $\mathcal{F}_1$ with an isomorphism $f: A \longrightarrow B$ and let a be the <-least (in T) element of $(A - B) \cup (B - A)$. Assume without loss of generality that

$a \in A \cap T(n)$ for some $n \in \omega$. Since $|A| = |B|$, using condition (i) in the definition of $\mathcal{F}_1$, $\exists!\ b \in ((B \cap T(n)) - A)$ such that $f(a) = b$ (since f preserves <). But condition (ii) and $\prec$-minimality of a imply $a \wedge b \in A \cap B$ with $a \wedge b \prec_p a$ and $a \wedge b \prec_q b$ for some distince $p, q \in \omega$. But $a \wedge b \prec_p a \rightarrow f(a \wedge b) = a \wedge b \prec_p f(a) = b$, giving a contradiction. □

Exactly in analogy to the definition of $\overline{\mathcal{C}}$ from $\mathcal{C}$, the general category $\overline{\mathcal{C}}_1 = \overline{\mathcal{F}}_1 \cup \overline{\mathcal{K}}_1$ is the class of structures isomorphic (by a unique canonical isomorphism) to some structure in $\mathcal{C}_1 = \mathcal{F}_1 \cup \mathcal{K}_1$. The morphisms of $\overline{\mathcal{C}}_1$ are defined by translating back and forth to canonical objects (as in the case of $\overline{\mathcal{C}}$).

Given $Z, W \in \overline{\mathcal{C}}_1$, $Z \subset\subset W$ means of course $Z \subseteq W$ and the natural inclusion map, $i_{ZW}: Z \rightarrow W$, is a morphism, $\langle Z, i_{ZW}, W \rangle \in \overline{\mathcal{C}}_1(Z, W)$. In analogy to Lemma 1.19 we have:

<u>Lemma 3.4</u> Given $Z \in \overline{\mathcal{C}}_1$, the finite subset $A \subseteq Z$ is a sub-object, $A \in \overline{\mathcal{F}}_1$ and $A \subset\subset Z$, iff A is a sub-structure of Z.

<u>proof</u>: ($\Rightarrow$) Clear. ($\Leftarrow$) We are assuming that the natural inclusion map is an isomorphic embedding, so it only remains to verify $A \in \overline{\mathcal{F}}_1$. We must construct the canonical isomorphism $\rho_A: A \longrightarrow \langle T; \sigma_1 \rangle$ from A to a

canonical image $\rho_A(A) \varepsilon \mathcal{F}_1$. The map ρ_A is defined by induction on the finite ordering $<^A$, beginning with $\rho_A(\text{root}(A)) = \text{root}(T)$. Given $a \varepsilon A$, let $n = |\{b \varepsilon A: b < a\}|$ and suppose $\rho_A(b)$ is defined for $b \varepsilon A, b < a$. Let $\rho_A(A)$ = the <-minimal member of $T(n)$ such that $(\forall b \varepsilon A) [b \prec a \rightarrow (\forall n \varepsilon \omega) (b \prec_n a \leftrightarrow \rho_A(b) \prec_n \rho_A(a))]$. It is easily checked that such a member of $T(n)$ exists and that $\rho_A: A \longrightarrow T$ is an isomorphic embedding with an image satisfying conditions (i), (ii), (iii) in the definition of $\mathcal{F}_1$. □

Suppose $Z \varepsilon \overline{\mathcal{K}}_1$ and $X \subseteq Z$ is infinite such that X is a substructure of Z and $(\forall x \varepsilon X) (\deg_X(x) = \deg_Z(x))$. In this case X is <u>not</u> necessarily a sub-object of Z. X may fail to be an object of $\overline{\mathcal{K}}_1$, due to an inappropriate interaction between the well ordering $<^X$ and the defined level ordering $<<^X$ (the definition is the usual one, $x <<^X y$ iff $|\{z \varepsilon X: z \quad x\}| < |\{z \varepsilon X: z \quad y\}|$). The objects $X \varepsilon \overline{\mathcal{K}}_1$ all satisfy $(\forall x,y \varepsilon X) (x <<^X y \rightarrow x < y)$ but the sub-structure Z may have $w,z \varepsilon Z$ such that $w <<^Z z$ and $z < w$. The simple lemma analogous to Lemma 1.20 fails.

<u>Lemma 3.5</u> Let $Y \varepsilon \overline{\mathcal{K}}_1$ and an infinite subset $Z \subseteq Y$ be given. Z is a sub-object of Y, $Z \varepsilon \overline{\mathcal{K}}_1$ and $Z \subset\subset Y$ iff

(i) Z is closed under $\wedge$ in Y

(ii) $(\forall z \in Z)\ (\deg_Z(z) = \deg_Y(z))$

(iii) $(\forall w,z \in Z)\ (w < z \rightarrow z \not<^Z w)$

proof: $(\Rightarrow)$ Clear. $(\Leftarrow)$ Conditions (i) (ii) show that the inclusion map is an isomorphic embedding which preserves degree so it only remains to show $Z \in \overline{\mathcal{K}}_1$. Condition (iii) allows an isomorphic embedding $\rho: Z \hookrightarrow T$ to be defined so that $\rho(Z(n)) \subseteq T(n)$ (where $Z(n) = \{z \in Z: |\{w \in Z: w < z\}| = n\}$, and thus with the obvious definition of ρ, the image is canonical. $\square$

Theorem 3.6 Given $A \in \overline{\mathcal{F}}_1$ and $Y \in \overline{\mathcal{K}}_1$ and a finite partition $c: \binom{Y}{A} \rightarrow r$ there exists $X \in \overline{\mathcal{K}}_1$ such that X reduces c (i.e., $X \subset\subset Y$ and c is constant on $\binom{X}{A}$).

Note that the notation $\binom{Y}{A}$, $\binom{X}{A}$ and $X \subset\subset Y$ is of course being used here in the context of $\overline{\mathcal{C}}_1$ rather than $\overline{\mathcal{C}}$.

Example 3.7 Consider again a partition of pairs from the infinite binary tree. Let $Y = \bigcup_{n \in \omega} {}^{n}2$ be structured by inheritance from $\langle T; \sigma_1 \rangle$ and let $c: [Y]^2 \longrightarrow r$ be a finite partition of the unordered pairs. In general a pair of nodes may not be closed under $\wedge$, so we classify the pair according to the isomorphism type (in the sense of $\mathcal{F}_1$) of its $\wedge$-closure. There are 4 isomorphism types which appear as $\wedge$-closures of a pair in the binary tree:

 $= \{0, \langle 0 \rangle\}$  $= \{0, \langle 1 \rangle\}$

 $= \wedge\text{-closure } \{\langle 0 \rangle, \langle 1,0 \rangle\}$
$= \{\langle 0 \rangle, \langle 1,0 \rangle, 0\}$

 $= \wedge\text{-closure } \{\langle 0,0 \rangle, \langle 1 \rangle\}$
$= \{\langle 0,0 \rangle, \langle 1 \rangle, 0\}$

Using 4 applications of Theorem 3.6 there is an $X \in \bar{\mathcal{K}}_1$, $X \subset\subset Y$ such that $|c''[X]^2| \leq 4$. Furthermore, if A is any one of the 4 finite canonical objects above, and $Z \in \bar{\mathcal{K}}_1$ is a sub-object, $Z \subset\subset Y$, then $\binom{Z}{A} \neq 0$ and all 4 isomorphism types are in this sense 'essential'.

<u>proof of 3.6</u> The method of proof involves translating the given partition $c: \binom{Y}{A} \longrightarrow r$, to a corresponding partition $\dot{c}: \binom{\dot{Y}}{\dot{A}} \longrightarrow r$ where $\dot{A} \in \bar{\mathcal{F}}$ and $\dot{Y} \in \bar{\mathcal{K}}$. Theorem 2.10 can be applied, and the result then translated back to the category $\bar{\mathcal{C}}_1$.

It is enough to prove the theorem in the case where A and Y are canonical, $A \in \mathcal{F}_1$ and $Y \in \mathcal{K}_1$. Let $\dot{A}$ = the $\wedge$-closure of A in T with structure inherited from $\langle T; \sigma \rangle$ so $\dot{A} \in \mathcal{F}$. Let $\dot{Y}$ = the structure Y extended to interpret $<<$ and Δ by inheritance from $\langle T; \sigma \rangle$ so $\dot{Y} \in \mathcal{K}$.

Now A is a substructure of $\dot{A}|\sigma_1$, so any isomorphic embedding $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ restricts to an isomorphic embedding $\dot{f}|A \in \binom{Y}{A}$. Hence the partition $\dot{c}: \binom{\dot{Y}}{\dot{A}} \rightarrow r$ is induced naturally from $\kappa: \binom{Y}{A} \to r$ by defining for any $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ $\dot{\kappa}(\dot{f}) = \kappa(\dot{f}|a)$.

We have $\dot{A} \in \hat{\mathcal{F}}$, $\dot{Y} \in \mathcal{K}$ with a finite partition $\dot{\kappa}: \binom{\dot{Y}}{\dot{A}} \to r$, so Theorem 2.10 gives $\dot{W} \in \overline{\mathcal{K}}$ such that $\dot{W} \subset\subset \dot{Y}$ (in the sense of $\overline{\mathcal{K}}$) and $\dot{\kappa}$ is constant on $\binom{\dot{W}}{\dot{A}}$. Put $W = \dot{W}|\sigma_1$. Let $\rho_{\dot{W}}$ be the canonical isomorphism from $\dot{W}$ to an object of $\mathcal{K}$, and note that $\rho_{\dot{W}}$ certainly preserves all the structure of $W = \dot{W}|\sigma_1$, and demonstrates that $W \in \overline{\mathcal{K}}_1$ and $W \subset\subset Y$.

To complete the proof, W must be reduced to some $X \in \overline{\mathcal{K}}_1$, $X \subset\subset W$ such that

$$f \in \binom{X}{A} \longrightarrow f = \dot{f}|A \text{ for some } \dot{f} \in \binom{\dot{W}}{\dot{A}}.$$

Having such an X, suppose $f, g \in \binom{X}{A}$ so $f = \dot{f}|A$ and $g = \dot{g}|A$ where $\dot{f}, \dot{g} \in \binom{\dot{W}}{\dot{A}}$. Since W reduces $\dot{\kappa}$, then $\dot{\kappa}(\dot{f}) = \dot{\kappa}(\dot{g})$ and this means $\kappa(f) = \kappa(g)$ so X reduces κ.

The $X \in \overline{\mathcal{K}}_1$ which satisfies * is defined inductively as a subset of W and inherits its structure from W. Let x_0 be the $\prec$-root of W. Having x_i for $i \in n$ let x_n be the $<$-least member of $\dot{W}(n)$ such that

<u>if</u> w is the least element of
$\{w \in W$: w is an immediate $\prec$-successor of some
x_k $(k \in n)$ and $(\forall k \in n)(w \not\preccurlyeq x_k)\}$
<u>then</u> $w \preccurlyeq x_n$.

It is easily verified that

$X = \{x_n : n \in \omega\} \in \overline{\mathcal{K}}_1$, $X \subset\subset W$,

$(\forall n \in \omega)(|X \cap \dot{W}(n)| = 1)$, and

$(\forall x \in X)\ (\forall w \in W - X)\ (w \prec x \to w \prec_0 x)$.

We prove that * holds by induction on $|A|$. It is trivial for $|A| = 1$. For $|A| = n$, let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and let $B = \{a_0, a_1, \ldots, a_{n-2}\}$ where $a_0 < a_1 < a_2 \ldots < a_{n-1}$. Put $\dot{B}$ = the Δ-closure of B in $\dot{A}$. Note that $a_0 <<^{\dot{A}} a_1 <<^{\dot{A}} \ldots\ <<^{\dot{A}} a_{n-1}$ and for some m, $0 < m < n$ $\dot{A} - \dot{B} = \{a_m \Delta a_{n-1}, a_{m+1} \Delta a_{n-1}, \ldots a_{n-1} \Delta a_{n-1}\}$. For $m \leqslant j < n-1$ we have $a_j \Delta a_{n-1} \prec_0 a_{n-1}$ since $a_j \Delta a_{n-1} \in W - X$.

Given $f \in \binom{X}{A}$ let $g = f \upharpoonright B$ and by induction assume $g = \dot{g} \upharpoonright B$ for some $\dot{g} \in \binom{\dot{W}}{\dot{B}}$. Define $\dot{f}(a)$ for $a \in \dot{A}$ by

$$\dot{f}(a) = \begin{cases} \dot{g}(a) & \text{if } a \in B \\ \dot{g}(a_j) \ \Delta\ f(a_{n-1}) & \text{if } a = a_j \ \Delta\ a_{n-1} \text{ where } m \leqslant j < n-1 \\ f(a_{n-1}) & \text{if } a = a_{n-1} \end{cases}$$

Clearly $\dot{f}: \dot{A} \longrightarrow \dot{W}$ is one-one (since $(\forall n \in \omega)(|X \cap W(n)| = 1)$) and the image $\dot{f}(A)$ = the Δ-closure in $\dot{W}$ of $f(A)$ (which is an object of $\overline{\mathcal{F}}$). For $a, b \in \dot{A}$ and $k \in \omega$, it is easily checked that $a \prec_k b \leftrightarrow \dot{f}(a) \prec_k \dot{f}(b)$ and since the full structure on objects in $\overline{\mathcal{F}}$ is definable from the $\prec_k$ $(k \in \omega)$, we conclude that $\dot{f} \in \binom{\dot{W}}{\dot{A}}$. By its definition $\dot{f} \upharpoonright A = f$ and hence X satisfies *. □

Corollary 3.8 Let $Y \in \overline{\mathcal{K}}_1$ and $A \in \overline{\mathcal{F}}_1$ and suppose $S \subseteq Y$ is a substructure of Y such that $(\forall s \in S)(\deg_S(s) = \deg_Y(s))$. Given any finite partition $\pi: \binom{S}{A} \longrightarrow r$ (where $\binom{S}{A}$ is the set of isomorphic embeddings of A into S) there exists a substructure $R \subseteq S$ such that $(\forall r \in R)(\deg_R(r) = \deg_S(r))$ and π is constant on $\binom{R}{A}$.

Proof: Using Lemma 3.5, the only reason that S may fail to be an object of $\overline{\mathcal{K}}_1$ is in case for some $s,t \in S$, $s \prec t$ but $t \prec^S s$. By an obvious inductive construction, we can find a substructure $S' \subseteq S$ such that $(\forall s \in S')(\deg_{S'}(s) = \deg_S(s) = \deg_Y(s))$ and $(\forall s,t \in S')(s \prec t \rightarrow t \not\prec^{S'} s)$. Hence $S' \in \overline{\mathcal{K}}_1$, $S' \subseteq\subseteq Y$ and the partition π restricted to $\binom{S'}{A}$ is exactly as in Theorem 3.6. The theorem gives $R \in \overline{\mathcal{K}}_1$ such that $R \subseteq\subset S' \subseteq\subseteq Y$ and π is constant on $\binom{R}{A}$. □

For the purpose of proving the next theorem we make a somewhat artificial definition of a category $\mathcal{K}_2$. The intuitive idea behind the objects of $\mathcal{K}_2$ is that they should look like subsets $S \subseteq X$ where $X \in \mathcal{K}_1$, and S satisfies $(\forall s \in S)(\exists x$ in the $\wedge$-closure of S such that $s \prec x$ & $\deg_S(x) = \deg_X(x) > 1)$. Here $\deg_S(x) = |\{t \in S: x \prec t$ & $(\forall s \in S)(s \prec t \rightarrow s \preccurlyeq x)\}|$. The objects of $\mathcal{K}_2$ will actually be much more highly structured (and in this sense, 'artificial') but it will be clear (by using a simple inductive construction) that any set S as above will contain in its $\wedge$-closure, an object from $\overline{\mathcal{K}}_2$.

<u>Definition 3.9</u> An object $X \in \mathcal{K}_2$ will be defined by first giving its <u>skeleton</u>, $\dot{X}$, which is an object of $\overline{\mathcal{K}}_1$ with $(\forall \dot{x} \in \dot{X})\ (\deg_{\dot{X}}(\dot{x}) > 1)$. Nodes of degree 1 are then attached to this skeleton. $\dot{X} \in \overline{\mathcal{K}}_1$ must satisfy,

(i) $\dot{X} \leq \langle T;\sigma_1\rangle$ with inherited structure

(ii) $(\forall n \in \omega)\ (|\dot{X} \cap T(2n)| = 1\ \&\ |\dot{X} \cap T(2n+1)| = 0)$

(iii) $(\forall t \in T - \dot{X})(\forall \dot{x} \in \dot{X})\ (t \prec \dot{x} \rightarrow t \prec_0 \dot{x})$

(iv) $(\forall \dot{x} \in \dot{X})\ (\deg_{\dot{X}}(\dot{x}) > 1)$

The object $X \in \mathcal{K}_2$ is

$X = \dot{X} \cup \{t \in T:$ for some $\dot{x} \in \dot{X}$, t is the immediate $\prec_0$-successor of $\dot{x}\}$

$= \dot{X} \cup \{\dot{x}\frown 0 : \dot{x} \in \dot{X}\}$, with structure inherited from $\langle T;\sigma_1\rangle$.

Note that $(\forall x \in X)\ (\deg_X(x) > 1 \leftrightarrow x \in \dot{X})$ and if $\deg_X(x) = 1$, then all the $\prec$-extensions of x are $\prec_0$-extensions and all the $\prec$-predecessors in $\dot{X}$ of x are $\prec_0$-predecessors. Although $\dot{X} \in \overline{\mathcal{K}}_1$, X itself fails to be an object of $\overline{\mathcal{K}}_1$, because the attachment of nodes to $\dot{X}$ only as $\prec_0$-extensions, has caused the condition $x \prec y \rightarrow y \not\prec x$ to be violated by X.

$\mathcal{F}$ ... $(\forall a$ ... $A)(deg_A(a) = 1 \rightarrow (\exists b \in A)(a \prec_0 b))$

$\mathcal{F}$ ... $\mathcal{K}$,

... $\mathcal{K}_2$, $Z \in \mathcal{C}_2$, a morphism from A to Z

... embedding, $f: A \hookrightarrow Z$ such that

... $\rightarrow deg_Z f(a) \leq 1)$. A morphism from

... morphic embedding $f: X \hookrightarrow Z$ such that

... $= deg_Z(f(x)))$.

... that composition of morphisms (as maps) is

... and the identity maps are morphisms so $\mathcal{C}_2$ is

...

... Every isomorphism between objects of $\mathcal{C}_2$ is

... map

... isomorphism between finite objects, this

... $\hat{\mathcal{F}} \subseteq \hat{\mathcal{F}}_1$, and Lemma 3.3. Infinite canonical

... $\hat{\mathcal{K}}$, are defined in T from their skeletons

... it's enough to show that distinct skeletons

... morphic. But $\dot{X} \in \overline{\mathcal{K}}_1$ is just a stretched out

... canonical $\sigma_{\dot{X}}(\dot{X}) \in \mathcal{K}_1$, and the conditions

... imply that $\dot{X}$ is the unique skeleton iso-

... $(\dot{X})$. Since distinct objects of $\mathcal{K}_1$ are not

... conclude that distinct skeletons are not

... □

<u>Definition 3.12</u> The general category $\overline{\mathcal{C}}_2 = \overline{\mathcal{F}}_2 \cup \overline{\mathcal{K}}_2$ is defined as usual via canonical isomorphisms to canonical objects in $\mathcal{C}_2 = \mathcal{F}_2 \cup \mathcal{K}_2$.

<u>Remark 3.13</u> An object $X \in \overline{\mathcal{K}}_2$ will have a skeleton $\dot{X} \in \overline{\mathcal{K}}_1$ which satisfies $\dot{X} = \{x \in X: \deg_X(x) > 1\}$ and $X = \dot{X} \cup \{x \in X:$ x is the immediate $\prec_0$-successor (in X) of some $\dot{x} \in \dot{X}\}$.

A special skeleton $\dot{X} \subseteq \langle T;\sigma_1\rangle$ as in the definition of a canonical object $X \in \mathcal{K}_2$ will be called a <u>canonical skeleton</u>.

<u>Lemma 3.14</u> Given $Z \in \overline{\mathcal{C}}_2$, the finite subset $A \subseteq Z$ is a sub-object, $A \in \overline{\mathcal{F}}_2$ and $A \subset\subset Z$, iff A is a substructure of Z and $(\forall a \in A)\ (\deg_A(a) \leqslant 1 \rightarrow \deg_Z(a) \leqslant 1)$.

<u>proof</u>: ($\Rightarrow$) Clear. ($\Leftarrow$) Since A is a substructure of Z, it is clear (as in the proof of Lemma 3.4) that $A \in \overline{\mathcal{F}}_1$. To show that $A \in \overline{\mathcal{F}}_2$, A must satisfy $(\forall a \in A)\ (\deg_A(a) = 1 \rightarrow (\exists b \in A)(a \prec_0 b))$. This follows from the assumption $(\forall a \in A)\ (\deg_A(a) \leqslant 1 \rightarrow \deg_Z(a) \leqslant 1)$ along with the fact that

$\deg_Z(a) = 1 \longrightarrow (\forall z \in Z)\ (a \prec z \longrightarrow a \prec_0 z)$ which holds for every $Z \in \overline{\mathcal{C}}_2$. $\square$

Lemma 3.15 Given $Z \in \overline{\mathcal{C}}_2$, the infinite subset $Y \subseteq Z$ is a sub-object, $Y \in \overline{\mathcal{K}}_2$ & $Y \subset\subset Z$, iff

(i) Y is a substructure of Z.

(ii) $(\forall y \in Y)(\deg_Y(y) = \deg_Z(y))$

(iii) $Y \in \overline{\mathcal{K}}_2$

Proof: Clear.

Theorem 3.16 Given $A \in \overline{\mathcal{F}}_2$ and $Y \in \overline{\mathcal{K}}_2$ and a finite partition $\pi : \binom{Y}{A} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_2$ such that X reduces π.

Example 3.17 We again consider a partition of pairs from the binary tree, but this time the binary tree is a substructure of an object in $\mathcal{K}_2$. Let $\dot{Y} \subseteq \langle T;\sigma_1\rangle$ be the unique canonical skeleton which is isomorphic to the binary tree, $\bigcup_{n\in\omega} {}^n 2 \in \mathcal{K}_1$, and let Y be the associated canonical object $Y = \dot{Y} \cup \{t \in T$: for some $\dot{y} \in \dot{Y}$, t is the immediate $\prec_0$ extension of $\dot{y}\}$. Put $S = \{y \in Y: \deg_Y(y) = 1\}$ and note $\langle S;\prec\rangle$ is a binary tree. Let $\pi: [S]^2 \longrightarrow r$ be a finite partition of the pairs from S. For any pair in S the $\wedge$-closure in Y is a finite sub-object of Y (in the sense of $\overline{\mathcal{F}}_2$) by Lemma 3.14 (since the $\wedge$-closure never adds nodes of degree $\leqslant 1$, and every $s \in S$ has degree 1, the degree condition on sub-objects is satisfied). The pairs in S can be classified according to the isomorphism type of their $\wedge$-closure in Y. There are three isomorphism types which occur as the closures of a pair from the binary tree $S \subseteq Y$.

$$\diagup = \{0, \langle 0 \rangle\}$$

$$\rangle = \wedge\text{-closure } \{\langle 0\rangle, \langle 1,0\rangle\} = \{\langle 0\rangle, \langle 1,0\rangle, 0\} \qquad \wedge = \wedge\text{-closure } \{\langle 0,0\rangle, \langle 1\rangle\} = \{\langle 0,0\rangle, \langle 1\rangle, 0\}$$

Note that $\diagdown = \{0, \langle 1\rangle\} \notin \hat{\mathcal{F}}_2$ since $0 \in \diagdown$ satisfies $\deg_{\diagdown}(0) = 1$ but there is no $b \in \diagdown$ such that $0 \prec_0 b$.

Using three applications of Theorem 3.16, there exists $X \in \overline{\mathcal{K}}_2$ such that the naturally induced partition is constant on $\binom{X}{\diagup}$, $\binom{X}{\rangle}$, and $\binom{X}{\wedge}$. Thus, with $P = \{x \in X: \deg_X(x) = 1\}$ we have $|c\,[P]^2| \leq 3$. Note $\langle P;\prec\rangle$ is a binary tree substructure of $\langle S;\prec\rangle$. If A is any one of the three finite objects above note that $\binom{X}{A} \neq 0$, so all three isomorphism types are represented as the closure of some pair from P. In this sense all three isomorphism types are essential.

<u>proof of 3.16</u> We can assume $A \in \mathcal{F}_2$ and $Y \in \mathcal{K}_2$. We will translate the given partition to a corresponding partition $\dot{c}:\binom{\dot{Y}}{\dot{A}} \to r$, where $\dot{Y} \in \overline{\mathcal{K}}_1$ is the skeleton of Y and $\dot{A} \in \hat{\mathcal{F}}_1$ is just A itself, but thought of in the context $\hat{\mathcal{F}}_1$ rather than $\hat{\mathcal{F}}_2$. Before defining $\dot{c}$, we need a fixed map which attaches the degree 1 nodes of Y to the skeleton, $\dot{Y}$. For $y \in Y - \dot{Y} = \{y \in Y: \deg_Y(y) = 1\}$, let $\dot{y}$ denote the $\prec_0$-immediate predecessor of y in Y.

Note that $y \mapsto \dot{y}$ is a bijection from $Y - \dot{Y}$ onto $\dot{Y}$, so typically an arbitrary member of $\dot{Y}$ will be denoted $\dot{y} \in \dot{Y}$ where $y \in Y - \dot{Y}$ is the corresponding node of degree 1. The specific highly organized structure of the objects $Y \in \mathcal{K}_2$ was designed so that y is the immediate $<$-successor of $\dot{y}$ (as well as the immediate $\prec_0$-successor and the immediate $\prec$-successor) and hence $(\forall \dot{y}, \dot{z} \in \dot{Y})\ (\dot{y} < \dot{z} \leftrightarrow \dot{y} < z \leftrightarrow y < \dot{z} \leftrightarrow y < z)$.

Given $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ the corresponding $f: A \hookrightarrow Y$ is defined for $a \in A$ by

$$f(a) = \begin{cases} \dot{f}(a) & \text{if } \deg_A(a) > 1 \\ y \text{ such that } \dot{y} = \dot{f}(a) & \text{if } \deg_A(a) \leqslant 1 \end{cases}$$

f is clearly an injection $f: A \hookrightarrow Y$ such that $(\forall a \in A)\ (\deg_A(a) \leqslant 1 \rightarrow \deg_Y(f(a)) = 1)$. To show that $f \in \binom{Y}{A}$, we must show that f is an isomorphic embedding of the σ_1-structure A into the σ_1-structure Y. Since the entire structure on A is definable from $\langle A; <, \prec_n \rangle_{n\in\omega}$, it suffices to show that f preserves $<$ and $\prec_n$ for $n \in \omega$. For distinct $a,b \in A$ let $y,z \in Y - \dot{Y}$ satisfy $\dot{y} = \dot{f}(a)$ and $\dot{z} = \dot{f}(b)$. We have $a < b \leftrightarrow \dot{f}(a) < \dot{f}(b) \leftrightarrow \dot{y} < z \leftrightarrow y < \dot{z} \leftrightarrow y < z$. Since $f(a)$ is either $\dot{y}$ or y and $f(b)$ is either $\dot{z}$ or z, f preserves $<$. Also $a \prec_n b \leftrightarrow \dot{f}(a) \prec_n \dot{f}(b) \leftrightarrow \dot{f}(a) \prec_n z$, so if $f(a) = \dot{f}(a)$ then $a \prec_n b \leftrightarrow f(a) \prec_n f(b)$. If $f(a) = y\ (\neq \dot{f}(a))$, then $\deg_A(a) \leqslant 1$, so by the definition of $\mathcal{F}_2$ and $\mathcal{K}_2$

$(\forall a,b \in A)\ (\forall n > 0)\ (a \not\prec_n b\ \ \&\ \ f(a) \not\prec_n f(b))$. Hence it only remains to show $a \prec_0 b \leftrightarrow f(a) \prec_0 f(b)$ where $f(a) = y$. But this is clear since the assumption $f(a) = y \neq z$ implies all $\prec$-predecessors of y in $Y - \dot{Y}$ are $\prec_0$-predecessors and all $\prec$-successors of y are $\prec_0$-successors so

$$a \prec_0 b \rightarrow \dot{f}(a) = \dot{y} \prec_0 y = f(a) \prec_0 \dot{f}(b) = \dot{z} \prec_0 z$$
$$\rightarrow f(a) \prec_0 f(b)$$

and $f(a) = y \prec_0 f(b) \rightarrow \dot{f}(a) = \dot{y} \prec_0 y \prec_0 \dot{f}(b) \rightarrow a \prec_0 b$.

Since $f \in \binom{Y}{A}$ we can define the induced partition $\dot{c}: \binom{\dot{Y}}{\dot{A}} \longrightarrow$ by $\dot{c}(\dot{f}) = c(f)$, where f is defined from $\dot{f}$ as above. Using Theorem 3.6, let $\dot{W} \in \overline{\mathcal{K}}_1$ satisfy $\dot{W} \subset\subset \dot{Y}$ and $\dot{c}$ is constant on $\binom{\dot{W}}{\dot{A}}$. Now we must define a substructure X of $\dot{W}$ such that $X \in \overline{\mathcal{K}}_2$, $X \subset\subset Y$ and for any $f \in \binom{X}{A}$ the corresponding $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ satisfies $\dot{f} \in \binom{\dot{W}}{\dot{A}}$.

The skeleton $\dot{X}$ of X is first constructed inside $\dot{W}$, exactly in analogy to the way a canonical skeleton is constructed in T. Define the levels of $\dot{W}$ as usual by $\dot{W}(n) = \{w \in \dot{W}: |\{x \in W: x \prec w\}| = n\}$. Let $\dot{X} \in \overline{\mathcal{K}}_1$ be the unique σ_1-structure which satisfies

(i) $\dot{X} \subseteq \dot{W}$ with inherited structure

(ii) $(\forall n \in \omega)\ (|\dot{X} \cap \dot{W}(2n)| = 1\ \ \&\ \ \dot{X} \cap \dot{W}(2n+1) = 0)$

(iii) $(\forall w \in \dot{W} - \dot{X})\ (\forall \dot{x} \in \dot{X})\ (w \prec \dot{x} \rightarrow w \prec_0 \dot{x})$

(iv) $(\forall \dot{x} \in \dot{X})\ (\deg_{\dot{X}}(\dot{x}) = \deg_{\dot{W}}(\dot{x}))$

Let $\overline{X} = \dot{X} \cup \{\dot{w} \in \dot{W}: \dot{w}$ is the immediate $\prec_0$-successor in $\dot{W}$ of some $\dot{x} \in \dot{X}\}$,

with structure inherited from Y. Clearly $\overline{X} \in \overline{\mathcal{K}}_2$ and $\overline{X}$ is a substructure of Y, but $\overline{X} \subseteq W$ and $(\forall \dot{w} \in \dot{W}) (\deg_{\dot{W}}(\dot{w}) = \deg_{\dot{Y}}(\dot{w}) = \deg_Y(\dot{w}) > 1)$ so $\overline{X}$ fails to be a sub-object of Y.

We emphasize now that the dot notation $y \mapsto \dot{y}$ is always being used as originally defined to denote the attachment bijection $Y - \dot{Y} \hookrightarrow \dot{Y}$, and it will not change meaning in the context of a new object of $\overline{\mathcal{K}}_2$ which has its own "skeleton attachment map".

Let $X = \dot{X} \cup \{y \in Y: \dot{y} \in \overline{X} - \dot{X}\}$ with structure inherited from Y, so X is just obtained by replacing the degree 1 nodes of $\overline{X}$ by their $\prec_0$-immediate successors in Y. It is easily seen that $X \in \overline{\mathcal{K}}_2$ (in fact X is isomorphic to $\overline{X}$), and $(\forall x \in X) (\deg_X(x) = \deg_Y(x))$ so by Lemma 3.15 $X \subset\subset Y$.

Let $f \in \binom{X}{A}$ and define $\dot{f}: \dot{A} \longrightarrow \overline{X} \subseteq \dot{W}$ for $a \in \dot{A}$ by

$$\dot{f}(a) = \begin{cases} f(a) & \text{if } \deg_A(a) > 1 \\ \dot{y} & \text{such that } y \doteq f(a) \text{ if } \deg_A(a) \leq 1 \end{cases}$$

With this definition of $\dot{f}$, note that

$$f(a) = \begin{cases} \dot{f}(a) & \text{if } \deg_A(a) > 1 \\ y & \text{such that } \dot{y} = \dot{f}(a) \text{ if } \deg_A(a) \leq 1 \end{cases}$$

so f and $\dot{f}$ are related exactly as they were previously in this proof. The verification that $\dot{f} \in \binom{\dot{W}}{\dot{A}}$ assuming $f \in \binom{X}{A}$ is just the reverse of the verification we have already seen that $f \in \binom{Y}{A}$ assuming $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$. Given $g \in \binom{X}{A}$, let $\dot{g} \in \binom{\dot{W}}{\dot{A}}$ correspond to g as above. Using the definition of $\dot{c}$ and the fact that $\dot{W}$ is homogeneous for $\dot{c}$ we have $c(f) = \dot{c}(\dot{f}) = \dot{c}(\dot{g}) = c(g)$. Hence $X \in \overline{\mathcal{K}}_2$ satisfies $X \subset\subset Y$ and X is homogeneous for c. □

The combinatorial Theorem 3.16 can be presented in a somewhat different form by focusing only on the nodes of degree ≤ 1. For $W,Z \in \overline{\mathcal{C}}_2$, any morphism $f: W \longrightarrow Z$ satisfies $(\forall w \in W)(\deg_W(w) \leq 1 \rightarrow \deg_Z(f(w)) \leq 1)$, so we could define as objects, $\{w \in W: \deg_W(w) \leq 1\}$, and let the morphisms just be restrictions of the morphisms of $\overline{\mathcal{C}}_2$. Since all $\prec$-extensions between nodes of degree ≤ 1 are $\prec_0$-extensions, the $\prec_n$ relations are not really relevant to the structure of $\{w \in W: \deg_W(w) \leq 1\}$, but in the absence of $\wedge$-closure for the set $\{w \in W: \deg_W(w) \leq 1\}$, various 'meet-type' relations would have to be introduced with definitions like

$\forall\ w,z \in \{w \in W: \deg_W(w) \leq 1\}$,

$w\ {}_2\wedge_3\ z$ iff $w \wedge z \prec_2 w$ & $w \wedge z \prec_3 z$ where $w \wedge z$ is the meet of w and z in W. The relevant structure on the object $\{w \in W: \deg_W(w) \leq 1\}$ is therefore the partial order $\prec$, the well order $<$, and the 'meet type' relations ${}_p\wedge_q$ where $p,q \in \omega$. The point is that there are many ways to present the same theorem. In the next category the intuitive structure on objects consists only of 'meet type' relations ${}_p\wedge_q$ and a well ordering -- there are no

$\prec$-extensions. Such objects can be represented as sets of pairwise $\prec$-incomparable nodes in T. As in the definition of $\mathcal{C}_2$, the objects of $\mathcal{C}_3$ will actually be much more highly structured than the basic intuitive idea of a set of pairwise incomparables. The definition of the objects of $\mathcal{C}_3 = \mathcal{F}_3 \cup \mathcal{K}_3$ will be analogous to the definition of $\mathcal{C}_2 = \mathcal{F}_2 \cup \mathcal{K}_2$, and the corresponding combinatorial theorem will be proved by the same technique used for Theorem 3.16.

Definition 3.18 An object $X \in \mathcal{K}_3$ will be defined by first giving its skeleton, $\dot{X}$, which is an object of $\overline{\mathcal{K}}_1$. Nodes of degree zero are then attached to this skeleton $\dot{X} \in \overline{\mathcal{K}}_1$ must satisfy:

(i) $\dot{X}$ is a substructure of $\langle T, \sigma_1 \rangle$

(ii) $(\forall n \in \omega)\,(|\dot{X} \cap T(3n)| = 1$ &

$\dot{X} \cap T(3n+1) = \dot{X} \cap T(3n+2) = 0)$

(iii) $(\forall t \in T - \dot{X})\,(\forall \dot{x} \in \dot{X})\,(t \prec x \rightarrow t \prec_0 x)$

(iv) $(\forall \dot{x} \in \dot{X})\,(\deg_{\dot{X}}(\dot{x}) > 1)$

The set of degree zero nodes attached to this skeleton will be denoted X' and

$X' = \{t \in T$: t is the immediate $\prec_1$-successor (in T) of the immediate $\prec_0$-successor (in T) of some $\dot{x} \in \dot{X}\}$

$= \{\dot{x}^\frown 01 : \dot{x} \in \dot{X}\}$.

$X = \wedge$-closure $(\dot{X} \cup X')$

$= \dot{X} \cup X' \cup \{\dot{x}^\frown 0 : \dot{x} \in \dot{X}\}$ with structure inherited from $\langle T; \sigma_1 \rangle$.

Remark 3.19 The intuitively important part of any $X \in \mathcal{K}_3$ is the set X' of pairwise $\prec$-incomparables. The nodes X' are carefully attached to the skeleton, $\dot{X} \in \overline{\mathcal{K}}_1$, for the purpose of translating partitions into the context of $\overline{\mathcal{C}}_1$. The nodes $X - (\dot{X} \cup X')$ are a technical necessity.

We note X has no nodes of degree one and $X' = \{x \in X: \deg_X(x) = 0\}$ Although $\dot{X} \in \overline{\mathcal{K}}_1$, X has nodes of degree zero and is certainly not an object of $\overline{\mathcal{K}}_1$.

Defintion 3.20 Let $\mathcal{F}_3 = \{A \in \mathcal{F}_1: (\forall a \in A)(\deg_A(a) \neq 1)\}$. Let $\mathcal{C}_3 = \mathcal{F}_3 \cup \mathcal{K}_3$. For $A \in \mathcal{F}_3$, $X \in \mathcal{K}_3$, $Z \in \mathcal{C}_3$ a morphism from A to Z is an isomorphic embedding, $f: A \hookrightarrow Z$, such that $(\forall a \in A)(\deg_A(a) = 0 \rightarrow \deg_Z f(a) = 0)$.

A morphism from X to Z is an isomorphic embedding, $f: X \hookrightarrow Z$, such that $(\forall x \in \dot{X} \cup X')(\deg_X(x) = \deg_Z(f(x))$.

Remark 3.21 The isomorphic embedding condition on a morphism, f, from $X \in \mathcal{K}_3$ to $Z \in \mathcal{C}_3$ implies $(\forall x \in \dot{X})\ (f(x) \in \dot{Z})$. The degree condition implies $(\forall x \in X')\ (f(x) \in Z')$ and f restricted to $\dot{X}$ is a morphism from $\dot{X}$ to $\dot{Z}$ in the sense of $\overline{\mathcal{K}}_1$.

The composition of morphisms (as maps) is a morphism and the identity maps are morphisms so $\mathcal{C}_3$ is a category.

<u>Lemma 3.22</u> Every isomorphism between objects of $\mathcal{C}_3$ is an identity map.

<u>proof</u>: Just like 30.6.

<u>Definition 3.23</u> The general category $\overline{\mathcal{C}}_3 = \overline{\mathcal{F}}_3 \cup \overline{\mathcal{K}}_3$ is defined via isomorphisms as usual. A general object $X \in \overline{\mathcal{K}}_3$ will have $X' = \{x \in X: \deg_X(x) = 0\}$ and the skeleton, $\dot{X}$, of X can be defined as

$\dot{X} = \{x \in X: (\exists y \in X')(y$ is the immediate $\prec_1$-successor (in X) of the immediate $\prec_0$-successor (in X) of $x)\}$.

The special $\dot{X} \subseteq \langle T;\sigma_1\rangle$ as in the definition of a canonical $X \in \mathcal{K}_3$ will be called a <u>canonical skeleton</u>.

<u>Lemma 3.24</u> Given $Z \in \overline{\mathcal{C}}_3$, the finite subset $A \subseteq Z$ is a sub-object, $A \in \overline{\mathcal{F}}_3$ and $A \subset\subset Z$, iff $(\forall a \in A)(\deg_A(a) \leq 1 \to \deg_Z(a) = 0)$.

<u>proof</u>: Routine -- similar to 3.14.

<u>Lemma 3.25</u> Given $Y \in \overline{\mathcal{K}}_3$, the infinite subset $Z \subseteq Y$ is a sub-object, $Z \in \overline{\mathcal{K}}_3$ and $Z \subset\subset Y$, iff

(i) Z is a substructure of Y

(ii) $(\forall z \in \dot{Z} \cup Z')(\deg_Z(z) = \deg_Y(z))$

(iii) $Z \in \overline{\mathcal{K}}_3$

proof: Clear. □

Theorem 3.26 Given $A \in \overline{\mathcal{F}}_3$ and $Y \in \overline{\mathcal{K}}_3$ and a finite partition $c : \binom{Y}{A} \to r$ there exists $X \in \overline{\mathcal{K}}_3$ such that X reduces c.

Example 3.27 Let $Y \in \mathcal{K}_3$ be the object whose skeleton is the binary tree. That is, let $\dot{Y} \leq \langle T; \sigma_1 \rangle$ be the unique substructure isomorphic to $\bigcup_{n \in \omega} {}^n 2 \subseteq \langle T; \sigma_1 \rangle$ which satisfies

$(\forall n \in \omega)\ (|\dot{Y} \cap T(3n)| = 1\ \ \&\ \ Y \cap T(3n+1) = Y \cap T(3n+2) = 0)$ and $(\forall t \in T - \dot{Y})\ (\forall \dot{y} \in \dot{Y})\ (t \prec \dot{y} \to t \prec_0 \dot{y})$. Let $Y \in \mathcal{K}_3$ be the object with skeleton $\dot{Y}$ and recall $Y' = \{y \in Y : \deg_Y(y) = 0\}$. Let $c : [Y']^2 \longrightarrow r$ be a finite partition of the pairs from Y'. For any pair in Y' the $\wedge$-closure in Y is a finite sub-object of Y by Lemma 3.24 (since the $\wedge$-closure only adds nodes of degree > 1). The pairs in Y' are thus classified according to the isomorphism type (in the sense of $\mathcal{F}_3$) of their closure in Y. There are two isomorphism types which appear as closures of a pair from Y' :

 $= \wedge\text{-closure } \{\langle 0\rangle , \langle 1,0\rangle\}$
$= \{\langle 0\rangle , \langle 1,0\rangle , 0\}$

 $= \wedge\text{-closure } \{\langle 0,0\rangle , \langle 1\rangle\}$
$= \{\langle 0,0\rangle , \langle 1\rangle , 0\}$

Note that / $= \{0, \langle 0\rangle\} \notin \mathcal{F}_3$ since $\deg_{/}(0) = 1$. Using 2 applications of Theorem 3.26, there exists $X \in \overline{\mathcal{K}}_3$, $X \subset\subset Y$ such that $|c''[X]^2| \leq 2$. By the definition of $X \subset\subset Y$, we note that the skeleton $\dot{X} \in \overline{\mathcal{K}}_1$ satisfies

$\dot{X} \subset\subset \dot{Y}$ so $\dot{X}$ is isomorphic to $\dot{Y}$ and X is isomorphic to Y. Hence for $A = \bigwedge$ or $A = \rangle$, $\binom{X}{A} \neq 0$ and both isomorphism types are essential.

<u>proof of 3.26</u> This proof will exactly parallel the proof of Theorem 3.16 so some of the details will be omitted. Assume $A \in \hat{\mathcal{F}}_3$ and $Y \in \mathcal{K}_3$ and let $\dot{Y} \in \overline{\mathcal{K}}_1$ be the skeleton of Y and let $\dot{A}$ be just A considered as an object of $\overline{\mathcal{F}}_1$. Given $y \in Y'$, the map $y \mapsto \dot{y}$ will denote the attachment of y to the skeleton,

$\dot{y}$ = the $\prec_0$-immediate predecessor (in Y) of the $\prec_1$-immediate predecessor (in Y) of y. Given $\dot{y},\dot{z} \in \dot{Y}$ note that $(\dot{y} < \dot{z} \leftrightarrow \dot{y} < z \leftrightarrow y < \dot{z} \leftrightarrow y < z)$.

Given $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ the corresponding $f\colon A \hookrightarrow Y$ is defined for $a \in A$ by

$$f(a) = \begin{cases} \dot{f}(a) & \text{if } \deg_A(a) > 1 \\ y \quad \text{such that} \quad \dot{y} = \dot{f}(a) & \text{if } \deg_A(a) = 0 \end{cases}$$

f is clearly an injection $f\colon A \hookrightarrow Y$ and $(\forall a \in A)(\deg_A(a) = 0 \rightarrow \deg_Y(f(a)) = 0)$. As in the proof of 3.16 $(\forall a,b \in A)$ $(a < b \leftrightarrow f(a) < f(b)$. Also $a \prec_n b \leftrightarrow \dot{f}(a) \prec_n \dot{f}(b) \leftrightarrow \dot{f}(a) \prec_n f(b)$ so if $f(a) = \dot{f}(a)$ then $a \prec_n b \leftrightarrow f(a) \prec_n f(b)$. If $f(a) \neq \dot{f}(a)$ (so $\deg_A(a) = 0$), then for any $n \in \omega$ and $b \in A$ $a \not\prec_n b$. Since $\deg_Y(f(a)) = 0$ also $f(a) \not\prec_n f(b)$ and thus $f \in \binom{Y}{A}$. We define $\dot{c}(\dot{f})$ by $\dot{c}(\dot{f}) = c(f)$. Let $\dot{W} \in \overline{\mathcal{K}}_1$ satisfy $\dot{W} \subset\subset \dot{Y}$ (in the sense of $\overline{\mathcal{K}}_1$) and

$\dot{\kappa}$ is constant on $\binom{\dot{W}}{\dot{A}}$. Let $\dot{X} \varepsilon \overline{\mathcal{K}}_1$ be the unique σ_1-structure which satisfies,

(i) $\dot{X}$ is a substructure of $\dot{W}$.

(ii) $(\forall n \varepsilon \omega)\ (|\dot{X} \cap \dot{W}(3n)| = 1\ \&$

$\dot{X} \cap \dot{W}(3n+1) = \dot{X} \cap \dot{W}(3n+2) = 0)$

(iii) $(\forall w \varepsilon \dot{W} - \dot{X})\ (\forall \dot{x} \varepsilon \dot{X})\ (w \prec \dot{x} \rightarrow w \prec_0 \dot{x})$

(iv) $(\forall \dot{x} \varepsilon \dot{X})\ (\deg_{\dot{X}}(\dot{x}) = \deg_{\dot{W}}(\dot{x}) = \deg_{\dot{Y}}(\dot{x}) > 1)$

Let $X' = \{y \varepsilon Y$: $\dot{y} \varepsilon \dot{W}$ is the $\prec_1$-immediate successor in $\dot{W}$ of the $\prec_0$-immediate successor in $\dot{W}$ of some $\dot{x} \varepsilon \dot{X}\}$.

Let $X = \wedge$-closure $(\dot{X} \cup X')$ with structure inherited from Y. it is easily verified (exactly in analogy to the proof of 3.16) that $X \varepsilon \overline{\mathcal{K}}_3$, $X \trianglelefteq\subseteq Y$ and κ is constant on $\binom{X}{A}$.

□

CHAPTER 4

GALVIN'S THEOREM $\eta \longrightarrow [\eta]^2_{<\omega/2}$ AND A GENERALIZATION

In this chapter we fix the object Y as in example 3.27 where $Y \in \mathcal{K}_3$ has the binary tree $\dot{Y} \in \overline{\mathcal{K}}_1$ as skeleton, and $c: [Y']^2 \longrightarrow r$ was a finite partition of the pairs of degree zero nodes. We found $X \in \overline{\mathcal{K}}_3$ such that $X \subset\subset Y$ and $|c''[X']^2| \leq 2$. We also fix the notation $Q = Y'$.

<u>Definition 4.1</u> Let $\circledless$ denote the binary relation defined for $p,q \in Q$ by $p \circledless q \leftrightarrow p \wedge q \prec_0 p$ (or equivalently $p \circledless q \leftrightarrow p \quad q \prec_1 q$).

<u>Lemma 4.2</u> Given any $X \in \overline{\mathcal{K}}_3$ if $X \subset\subset Y$ then $\langle X'; \circledless \rangle$ is a countable dense linear order without endpoints -- an order type η set.

<u>proof</u>: Note that $X \subset\subset Y$ implies $\dot{X} \subset\subset \dot{Y}$, and since $\dot{Y}$ is binary, in fact $\dot{X}$ is isomorphic to $\dot{Y}$ and X is isomorphic to Y. It is enough to show that $\langle Q; \circledless \rangle$ is a set with $\circledless$-order type η.

Given $p \circledless q \circledless r$ where $p,q,r \in Q$, either $p \wedge q \prec q \wedge r$ or $q \wedge r \prec p \wedge q$. It is easily seen that in the first case $p \wedge q = p \wedge r \prec_0 p$ and in the second case $q \wedge r = p \wedge r \prec_1 r$. Hence $p \circledless r$ and $\langle Q; \circledless \rangle$ is a countable linear ordering. Given $p \in Q$ attached to $\dot{p} \in \dot{Y}$, find $\dot{y}, \dot{z} \in \dot{Y}$ such that $\dot{p} \prec_0 \dot{y}$ and $\dot{p} \prec_1 \dot{z}$.

Then $y \circledS p \circledS z$ (where $y \in Q$ is attached to $\dot{y}$ and $z \in Q$ is attached to $\dot{z}$), so $\langle Q; \circledS \rangle$ does not have endpoints. Given $p,q \in Q$, assume $p \circledS q$. If $\dot{q} \prec \dot{p}$ then let $\dot{z} \in \dot{Y}$ satisfy $\dot{p} \prec_1 \dot{z}$ so $z \wedge q = \dot{p}$ and $p \circledS z \circledS q$. Otherwise $p \wedge q \in \dot{Y}$, say $p \wedge q = \dot{y}$, and we pick $\dot{z} \in \dot{Y}$ such that $\dot{q} \prec_0 \dot{z}$. Since $\dot{y} \prec_1 q$, $\dot{y} \neq \dot{q}$ and $\dot{y} \prec_1 \dot{q} \prec_0 z$. Thus $p \circledS z \circledS q$ and the ordering $\langle Q; \circledS \rangle$ is dense. □

We would like to identify sub-objects of Y precisely with the order type η subsets of Q, but due to the highly organized structure of objects $X \in \overline{\mathcal{K}}_3$, the converse of Lemma 4.2 fails and we have only the following 'partial converse'.

<u>Lemma 4.3</u> Let $S \subseteq Q$. If S has $\circledS$-order type η then S contains a subset P such that $P = X'$ for some $X \subset\subset Y$, $X \in \overline{\mathcal{K}}_3$.

<u>proof</u>: Let $\overline{S} = \{y \in Y$: both $\{s \in S: y \prec_0 s\}$ and $\{s \in S: y \prec_1 s\}$ are densely ordered by $\circledS\}$. We claim that $\overline{S}$ is a substructure of $\dot{Y}$, satisfying $(\forall \bar{s} \in \overline{S})(\deg_{\overline{S}}(\bar{s}) = \deg_{\dot{Y}}(\bar{s}) = 2)$. We must show that $\overline{S} \subseteq \dot{Y}$ is non-empty and $(\forall \bar{s} \in \overline{S})(\exists \bar{u},\bar{v} \in \overline{S})$ such that $\bar{s} \prec_0 \bar{u}$ and $\bar{s} \prec_1 \bar{v}$. Let $\bar{s}$ be the $\prec$-least node of the $\wedge$-closure of S and put $U = \{t \in S: \bar{s} \prec_0 t\}$ and $V = \{t \in S: \bar{s} \prec_1 t\}$. Then

U and V are non-empty, $U \cup V = S$, and $(\forall u \in U)(\forall v \in V)(u \oslash v)$. Since S has $\oslash$-order type η, U and V are densely ordered by $\oslash$ and $\bar{S}$ is non-empty since $\bar{s} \in \bar{S}$. Given any $\bar{s} \in \bar{S}$ put $U = \{t \in S: \bar{s} \prec_0 t\}$ and let

$U' = U - \{u \in U: u$ is an endpoint of the dense linear order $\langle U; \oslash \rangle\}$.

Let $\bar{u}$ be the $\prec$-least node of the $\wedge$-closure of U'. As above, both $\{w \in U': \bar{u} \prec_0 w\}$ and $\{w \in U': \bar{u} \prec_1 w\}$ are densely ordered by $\oslash$. Because the only points deleted from U to form U' were endpoints, $\{w \in U: \bar{u} \prec_0 w\}$ and $\{w \in U: \bar{u} \prec_1 w\}$ are also both densely ordered by $\oslash$. Hence $\bar{u} \in \bar{S}$ and $\bar{s} \prec_0 \bar{u}$. By the same argument, $\bar{v} \in \bar{S}$ is found such that $\bar{s} \prec_1 \bar{v}$, and the claim is established.

Although $\bar{S}$ is a substructure of $\dot{Y}$ with all nodes of degree 2, $\bar{S}$ may fail to be a sub-object of $\dot{Y} \in \bar{\mathcal{K}}_1$ due to an inappropriate interaction between the level structure of $\bar{S}$ (defined from $\prec^{\bar{S}}$) and the well ordering, $<^{\bar{S}}$. The substructure $\dot{W} \subseteq \bar{S}$ of $\langle \bar{S}; \sigma_1 \rangle$ is defined by an obvious inductive construction so that $\dot{W} \in \bar{\mathcal{K}}_1$ and all nodes have degree 2, so $\dot{W} \subset\subset \dot{Y}$. If $\phi: \bar{S} \longrightarrow S$ is a fixed injection which satisfies $(\forall w \in \bar{S})(w \prec \phi(w))$, then it is easy to carry out the inductive construction of $\dot{W}$ so that $\dot{W}$ also satisfies

$(\forall v,w \in \dot{W})(w < v \leftrightarrow \phi(w) < v \leftrightarrow w < \phi(v) \leftrightarrow \phi(w) < \phi(v))$.

Let $\dot{X} \in \overline{\mathcal{K}}_1$ be the unique σ_1-structure which satisfies

(i) $\dot{X}$ is a substructure of $\dot{W}$

(ii) $(\forall n \in \omega)\ (|\dot{X} \cap \dot{W}(3n)| = 1\ \ \&\ \ X \quad W(3n+1) = X \quad W(3n+2) = 0)$

(iii) $(\forall w \in \dot{W} - \dot{X})(\forall \dot{x} \in \dot{X})(w \prec \dot{x} \rightarrow w \prec_0 \dot{x})$

(iv) $(\forall \dot{x} \in \dot{X})(\deg_{\dot{X}}(\dot{x}) = \deg_{\dot{W}}(\dot{x}))$

Let $X' = \{\phi(w)$: w is the $\prec_1$-immediate successor in $\dot{W}$ of the $\prec_0$-immediate successor in $\dot{W}$ of some $\dot{x} \in \dot{X}\}$. Let $X = \wedge$-closure $(\dot{X} \cup X')$ with structure inherited from Y. This construction of X is essentially the same as the construction at the end of the proof of Theorem 3.25, and it is easily checked that $X \in \overline{\mathcal{K}}_3$ with skeleton $\dot{X}$. X satisfies $X \subset\subset Y$ and $X' \subseteq S$. [

For any finite partition $c: [Q]^2 \longrightarrow r$ we found $X \in \overline{\mathcal{K}}_3$ such that $X \subset\subset Y$ and $c''[X']^2 \leqslant 2$. Using Lemma 4.2, this result translates to the assertion that there exists a subset $P \subseteq Q$ of ⓢ-order type η such that $c\ [P]^2 \leqslant 2$. We also found that both isomorphism types  and  are represented as the closure of pairs from X' when $X \in \overline{\mathcal{K}}_3$ is any sub-object $X \subset\subset Y$. Using Lemma 4.3, this result translates to the assertion that there is a partition $c: [Q]^2 \longrightarrow 2$ such that for every subset $P \subseteq Q$, if P has ⓢ order type η , then $|c'': [P]^2| = 2$. This is an unpublished result of Fred Galvin which he proved in 1969. It is usually stated using some variant of the standard partition arrow notation.

<u>Theorem 4.4</u> (F. Galvin 1969)

$$\eta \longrightarrow (\eta)^2_{<\omega/2} \quad \text{and} \quad \eta \not\longrightarrow (\eta)^2_{<\omega/1} .$$

<u>Definition 3.31</u> When β and γ represent order types and $n, c \le \omega$ the notation $\beta \to (\gamma)^n_{<\omega/c}$ means that for any ordered set Q of order type β and any partition $\pi : [Q]^n \longrightarrow r$ of the n-tuples of Q with $r < \omega$ colors, there is a subset $P \subseteq Q$ of order type γ such that $|\pi''[P]^n| \le c$. The slash through the arrow indicates the negation of this statement.

<u>Remark 3.32</u> Thinking in terms of the category $\bar{\mathcal{C}}_3$ we see that $\eta \not\longrightarrow (\eta)^2_{<\omega/1}$ because there are two isomorphism types of closures of pairs from Q. It is simpler to say that there are two possible interactions in a pair between the well ordering $<^Q$ on Q and the η-type order $\odot^Q$ on Q. Applying this reasoning to the case of an n-tuple, there are n! possible interactions in an n-tuple between the orderings $<$ and $\odot$. But if we think of the order type η-set Q as the rational numbers and look at an arbitrary n-tuple $a_0 \odot a_1 \odot \ldots \odot a_{n-1}$, the relative distances between $\odot$-adjacent members of the n-tuple also provides a means of classification and there are $(n-1)!$ distinct possibilities here. Thus we have classified $n!(n-1)!$ distinct types of n-tuple from which a natural partition $\pi : Q^n \longrightarrow n!(n-1)!$ can be defined

by mapping the distinct types of n-tuple to distinct colors from $n!(n-1)!$. It is easily checked that each of these $n!(n-1)!$ types of n-tuple is essential to any $P \subseteq Q$ of order type η and therefore the above partition $c:[Q]^n \longrightarrow n!(n-1)!$ shows that $\eta \not\longrightarrow (\eta)^n_{<\omega/n!(n-1)!-1}$.

Along with proving the theorem $\eta \longrightarrow (\eta)^2_{<\omega/2}$, Galvin conjectured results for higher exponents etc. Among these was the conjecture $\eta \longrightarrow (\eta)^n_{<\omega/n!(n-1)!}$. Richard Laver (in late 1969) translated one of these conjectures into a tree theorem (essentially Theorem 1.3) which he then proved (without knowledge of the Halperin-Lauchli result by a long and messy argument -- unpublished). Consequently Laver showed that

$$\begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix} \longrightarrow \begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix}^{1,1,\ldots,1}_{<\omega/d!}$$

where d is the number of η's in each column.

Given positive integers d, $k_0, k_1, \ldots, k_{d-1}, q$ the notation

$$\begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix} \longrightarrow \begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix}^{k_1,k_2,\ldots,k_{d-1}}_{<\omega/q}$$

means that for any 'd-vector'

$$\begin{pmatrix}Q_0\\ Q_1\\ \vdots\\ Q_{d-1}\end{pmatrix}$$

of d disjoint order type η sets, Q_i, $(i \in d)$

and any partition

$\kappa : \{C \subseteq \bigcup_{i \in d} Q_i : (\forall i \in d)(|C \cap Q_i| = k_i)\} \longrightarrow r$ where $r < \omega$,

there exists a d-vector

$$\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_{d-1} \end{pmatrix}$$ of order type η sets $P_i \subseteq Q_i$ $(i \in d)$

such that $|\kappa''\{C \subseteq \bigcup_{i \in d} P_i : (\forall i \in d)(|C \cap P_i| = k_i)\}| \leq q$.

It was thought that the same ideas used for the result

$$\begin{pmatrix} \eta \\ \eta \\ \vdots \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} \eta \\ \eta \\ \vdots \\ \eta \end{pmatrix}^{1,1,\ldots,1}_{<\omega \; d!}$$ where d is the number of η's in each column

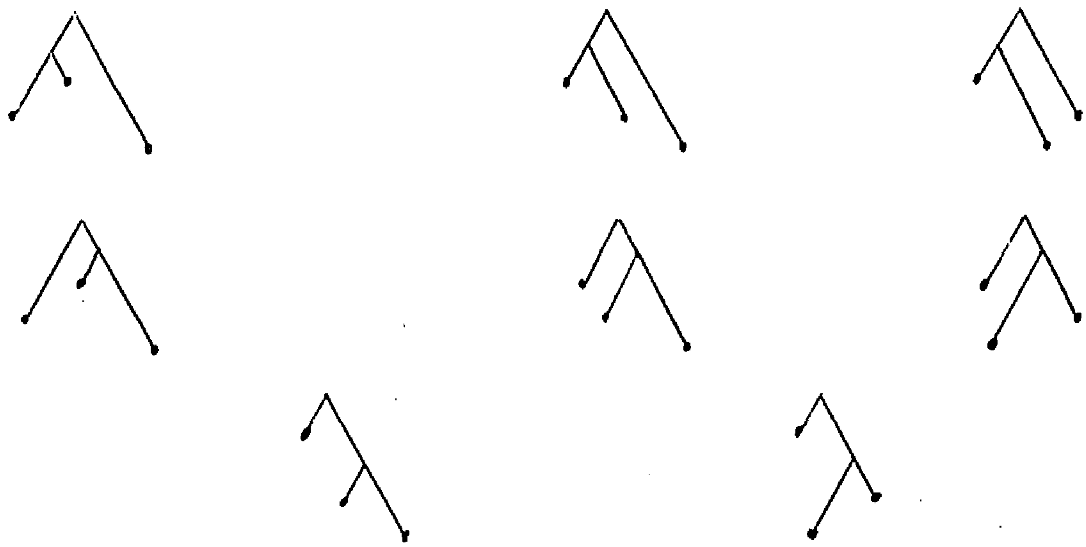
could be extended to prove Galvin's more general conjectures but such an argument was never carefully written down and in fact Galvin's conjecture $\eta \longrightarrow (\eta)^n_{<\omega/n!(n-1)!}$ is false. For example, we will show (by translating to the category $\overline{\mathcal{C}}_3$ and using Theorem 3.26) that $\eta \longrightarrow (\eta)^3_{<\omega/16}$ and $\eta \not\longrightarrow (\eta)^3_{<\omega/15}$. Before doing this, the pictorial notation for representing the isomorphism type of finite objects of $\overline{\mathcal{C}}_3$ is first simplified. The proof of $\eta \longrightarrow (\eta)^2_{<\omega/2}$ & $\eta \not\longrightarrow (\eta)^2_{<\omega/1}$ involved looking at the isomorphism types (of closures of pairs)

 $\wedge$-closure $\{\langle 0\rangle, \langle 1,0\rangle\}$ and

 $\wedge$-closure $\{\langle 0,0\rangle, \langle 1\rangle\}$.

The object  $\in \mathcal{F}_3$,  $=$ $\wedge$-closure $\{\langle 0\rangle, \langle 1,0\rangle\} =$ $\{\langle 0\rangle, \langle 1,0\rangle, 0\}$ is definable knowing only the 'meet type' of pairs of degree zero nodes and knowing the $<$-well ordering on the nodes. This information is simply represented pictorially by  where the dots are used only to represent the degree zero nodes, the vertex represents the meet (and thus the meet type), and the $<$-well ordering is represented by the vertical level relationships in the picture.

The following 8 objects and their mirror images gives a catalogue of all the isomorphism types which are realized as the $\wedge$-closure of a triple of nodes from $Q = Y - \acute{Y}$.



For example  $\varepsilon\ \hat{\mathcal{F}}_3$ denotes the $\wedge$-closure of $\{\langle 0\rangle, \langle 1,0,0,0\rangle, \langle 1,0,1\rangle\} = \{\langle 0\rangle, \langle 1,0,0,0\rangle, \langle 1,0,1\rangle, 0, \langle 1,0\rangle\}$, (where the closure is of course computed in $\langle T;\sigma_1\rangle$). It is clear that the 8 objects above along with their mirror images constitute all the canonical objects $A\ \varepsilon\ \hat{\mathcal{F}}_3$ such that

$(\forall a,b\ \varepsilon\ A)(a \prec b \ \rightarrow a \prec_0 b \text{ or } a \prec_1 b)$ and

$|\{a\ \varepsilon\ A\colon \deg_A(a) = 0\}| = 3.$

Hence by Lemma 3.24, the isomorphism type of the $\wedge$-closure of any triple from Q is represented by one of these objects. Let $A \in \mathcal{F}_3$ be one of these objects. To show that $\binom{Y}{A} \neq 0$, let $\dot{A}$ denote A considered in the context $\mathcal{F}_1$ and note that clearly $\binom{\dot{Y}}{\dot{A}} \neq 0$ (since $\langle \dot{Y}; \prec \rangle$ is the binary tree). Given an isomorphic embedding $\dot{f} \in \binom{\dot{Y}}{\dot{A}}$ define $f \in \binom{Y}{A}$ exactly as in the proof of Theorem 3.26

$$f(a) = \begin{cases} \dot{f}(a) & \text{if } \deg_A(a) > 1 \\ y & \text{such that } \acute{y} = \dot{f}(a) \text{ if } \deg_A(a) = 0. \end{cases}$$

Since any $X \in \mathcal{K}_3$ such that $X \subset\subset Y$ is isomorphic to Y, this shows $\binom{X}{A} \neq 0$, and in this sense each of the 16 objects is essential. Using Lemma 4.3 this translates as $\eta \not\rightarrow (\eta)^3_{<\omega/15}$. The assertion $\eta \longrightarrow (\eta)^3_{<\omega/16}$ follows from the above discussion and 16 applications of Theorem 3.26 followed by Lemma 4.2.

Note that the analysis of the type of a triple based only on the interaction between the ordering $<$ and $\circledless$, and on the ordering of distances between $\circledless$ -adjacent pairs fails to distinguish between

 and  (and likewise for mirror images)

 and  (and likewise for mirror images).

If the degree zero nodes are labelled p,q,r in the order $p \circledless q \circledless r$, the distinction between these pairs of isomorphism types can be described as depending on whether or not $(\exists z \in Q)(q \circledless z \circledless r \;\&\; z \circledless q \;\&\; z \circledless r \;\&\; p \circledless z)$.

<u>Theorem 4.7</u> $\eta \longrightarrow (\eta)^n_{<\omega/\phi(n)}$ and $\eta \not\longrightarrow (\eta)^n_{<\omega/\phi(n)-1}$ where $\phi(n)$ is defined by recursion beginning with $\phi(1) = 1$ and then $\phi(n) = \sum_{\ell=1}^{n-1} \binom{2n-2}{2\ell-1}\phi(\ell)\cdot\phi(n-\ell)$. Here $\binom{2n-2}{2\ -1}$ denotes the standard binomial coefficient .

Note $\phi(2) = \binom{2}{1}\phi(1)\ \phi(1) = 2$

$\phi(3) = \binom{4}{1}\ \phi(1)\ \phi(2) + \binom{4}{3}\ \phi(2)\ \phi(1) = 4\cdot1\cdot2 + 4\cdot2\cdot1 = 16$

<u>proof</u>: Following the analysis of $\eta \longrightarrow (\eta)^3_{<\omega/16}$ and $\eta \longrightarrow (\eta)^3_{<\omega/15}$, the problem is simply that of counting the number, $\phi(n)$, of essential types of n-tuples -- i.e.,

the number of canonical structures $A \varepsilon \hat{\mathcal{F}}_3$ such that $|\{a \varepsilon A: \deg_A(a) = 0\}| = n$ and $(\forall a,b \varepsilon A)((a \prec b \to (a \prec_0 b \text{ or } a \prec_1 b))$.

Given such an A, let $b \varepsilon A$ be the $\prec$-minimal node (the root of A) and put $A_0 = \{a \varepsilon A: b \prec_0 a\}$ and $A_1 = \{a \varepsilon A: b \prec_1 a\}$. Let $\ell = |\{a \varepsilon A_0: \deg_{A_0}(a) = 0\}|$ and $r = |\{a \varepsilon A_1: \deg_{A_1}(a) = 0\}|$ so $\ell + r = n$. When structured by inheritance, $A_0, A_1 \varepsilon \hat{\mathcal{F}}_3$ and for fixed $0 < \ell, r < n$ such that $\ell + r = n$ there are $\phi(\ell)\cdot\phi(r)$ possibilities for the isomorphism type of the pair $\langle A_0, A_1\rangle$. We have $A = A_0 \cup A_1 \cup \{b\}$ and the structure A is completely definable from the isomorphism type of the structures A_0 and A_1 along with the well ordering $<^A$. The well ordering $<^A$ interlaces the well orderings $<^{A_0}$ and $<^{A_1}$ and adjoins b as the $<$-minimal node of A. The number of possible ways $<^{A_0}$ and $<^{A_1}$ can be interlaced to form $<^A$ is the binomial coefficient

$$\binom{|A_0| + |A_1|}{|A_0|} = \binom{|A| - 1}{|A_0|}$$

and each of these possibilities corresponds to a different $<$-structure on $A_0 \cup A_1 \cup \{b\}$. It is easy to check by induction on n that $|A| = 2n - 1$ (where remember $n = |\{a \varepsilon A: \deg_A(a) = 0\}|$) and hence the binomial coefficient is $\binom{2n-2}{2\ell-1}$. The total number of possibilities for $A \varepsilon \hat{\mathcal{F}}_3$ satisfying $n = |\{a \varepsilon A: \deg_A(a) = 0\}|$ and

$(\forall a,b \in A)((a \prec b \rightarrow (a \prec_0 b \text{ or } a \prec_1 b))$ is therefore

$\sum_{\ell=1}^{n-1} 2^{2n-2}-1\ \phi(\ell)\cdot\phi(n-\ell)$ □

Before proving the polarized partition theorem

$$\begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix} \longrightarrow \begin{pmatrix}\eta\\ \eta\\ \vdots\\ \eta\end{pmatrix}^{1,1,\ldots,1}_{<\omega/n!}$$ where n is the number of η's in each column

and its generalization, it is necessary to upgrade Theorem 3.26 so that it deals with a partition $\mathcal{c}:\begin{pmatrix}Y, & Y\upharpoonright 1\\ A, & A\upharpoonright 1\end{pmatrix} \longrightarrow r$. Theorem 3.26 rests on Theorem 3.6 (the partition theorem for $\bar{\mathcal{C}}_1$) and we must begin by upgrading this theorem. For objects in the various categories, the notation $A\upharpoonright 1$ (or $X\upharpoonright 1$ etc.) will denote as usual, the singleton set consisting of the $\prec$-root of A.

<u>Theorem 4.8</u> Given $A \in \bar{\mathcal{F}}_1$ and $Y \in \bar{\mathcal{K}}_1$ and a finite partition $\mathcal{c}:\begin{pmatrix}Y, & Y\upharpoonright 1\\ A, & A\upharpoonright 1\end{pmatrix} \longrightarrow r$ there exists $X \in \bar{\mathcal{K}}_1$ such that $(X, X\upharpoonright 1)$ reduces $\mathcal{c}$.

<u>proof</u>: In the proof of Theorem 3.6 objects $\dot{A},\dot{Y} \in \bar{\mathcal{C}}$ corresponding to $A,Y \in \bar{\mathcal{C}}_1$ were defined so that the given partition $\mathcal{c}:\begin{pmatrix}Y\\ A\end{pmatrix} \rightarrow r$ translated ~~to a partition~~ $\dot{\mathcal{c}}:\begin{pmatrix}\dot{Y}\\ \dot{A}\end{pmatrix} \longrightarrow r$ and then Theorem 2.10 was applied. Looking more closely at this translation, it is apparent that it

entails a translation of any partition $\mathfrak{c}:\begin{pmatrix} Y, & Y\upharpoonright 1 \\ A, & A\upharpoonright 1 \end{pmatrix} \longrightarrow r$ to a partition $\dot{\mathfrak{c}}:\begin{pmatrix} \dot{Y}, & \dot{Y}\upharpoonright 1 \\ \dot{A}, & \dot{A}\upharpoonright 1 \end{pmatrix} \longrightarrow r$ and Theorem 2.11 can be applied to give $\dot{W} \in \overline{\mathcal{K}}$ such that $(\dot{W}, \dot{W}\upharpoonright 1)$ reduces $\dot{\mathfrak{c}}$. With $W = \dot{W}\upharpoonright\sigma_1 \in \overline{\mathcal{K}}_1$, the specific X constructed in proof 3.6 such that $X \in \overline{\mathcal{K}}_1$, $X \subset\subset W$ in fact satisfies $X\upharpoonright 1 = W\upharpoonright 1 = \dot{W}\upharpoonright 1 = \dot{Y}\upharpoonright 1 = Y\upharpoonright 1$ and hence $(X, X\upharpoonright 1)$ reduces $\mathfrak{c}$. □

<u>Theorem 4.9</u> Given $A \in \overline{\mathcal{F}}_3$ and $Y \in \overline{\mathcal{K}}_3$ and a finite partition $\mathfrak{c}:\begin{pmatrix} Y, & Y\upharpoonright 1 \\ A, & A\upharpoonright 1 \end{pmatrix} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_3$ such that $(X, X\upharpoonright 1)$ reduces $\mathfrak{c}$.

<u>proof</u>: Reread proof 3.26. As above, the translation from $\mathfrak{c}:\binom{Y}{A} \longrightarrow r$ to $\dot{\mathfrak{c}}:\binom{\dot{Y}}{\dot{A}} \longrightarrow r$ entails the translation from $\mathfrak{c}:\begin{pmatrix} Y, & Y\upharpoonright 1 \\ A, & A\upharpoonright 1 \end{pmatrix} \longrightarrow r$ to $\dot{\mathfrak{c}}:\begin{pmatrix} \dot{Y}, & \dot{Y}\upharpoonright 1 \\ \dot{A}, & \dot{A}\upharpoonright 1 \end{pmatrix} \longrightarrow r$ (where $\dot{A}, \dot{Y} \in \overline{\mathcal{C}}_1$ correspond to $A, Y \in \overline{\mathcal{C}}_3$). The combinatorial Theorem 4.8 gives $\dot{W} \in \overline{\mathcal{K}}_1$ satisfying $(\dot{W}, \dot{W}\upharpoonright 1)$ reduces $\dot{\mathfrak{c}}$. The construction in Theorem 3.26 of X from $\dot{W}$ yields $X\upharpoonright 1 = \dot{W}\upharpoonright 1 = \dot{Y}\upharpoonright 1 = Y\upharpoonright 1$ and hence $(X, X\upharpoonright 1)$ reduces $\mathfrak{c}$. □

For the sake of completeness we mention:

<u>Theorem 4.10</u> Given $A \in \overline{\mathcal{F}}_2$ and $Y \in \overline{\mathcal{K}}_2$ and a finite partition $\mathfrak{c}:\begin{pmatrix} Y, & Y\upharpoonright 1 \\ A, & A\upharpoonright 1 \end{pmatrix} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_2$ such that $(X, X\upharpoonright 1)$ reduces $\mathfrak{c}$.

proof: Reread proof 3.16 with modifications as above. □

Definition 4.11 For a positive integer d define $Y_d \in \mathcal{K}_3$ as the (unique canonical) structure whose (canonical) skeleton, $\dot{Y}_d$, satisfies

(i) if $b \in \dot{Y}_d$ is the root (the $\prec$-minimal node) then $\deg_{\dot{Y}_d}(b) = d$

(ii) if $y \in \dot{Y}_d$ is not the root then $\deg_{\dot{Y}_d}(y) = 2$.

For fixed $X \in \bar{\mathcal{K}}_3$, let b denote the root of X, and as usual $\dot{X}$ denotes the skeleton. For each $i \in \deg_X(b)$ define $P_i = \{x \in X\colon \deg_X(x) = 0 \text{ and } b \prec_i x\}$, so $X' = \bigcup_{i \in \deg_X(b)} P_i$. In particular for a fixed $d \in \omega$ and object $Y_d \in \mathcal{K}_3$ we denote $\{y \in Y_d\colon \deg_{Y_d}(y) = 0 \;\&\; \mathrm{root}(Y_d) \prec_i y\}$ as Q_i.

Lemma 4.12 Given any $X \in \bar{\mathcal{K}}_3$, and fixed $d \in \omega$, if $(X, X{\upharpoonright}1) \subset\subset (Y_d, Y_d{\upharpoonright}1)$ then for $i \in d$ the sets $\langle P_i; \circledcirc \rangle$ are disjoint sets of order type η and $P_i \subseteq Q_i$.

proof: Clear from Lemma 4.2 and its proof. □

Lemma 4.13 For $i \in d$ let $S_i \subseteq Q_i$. If each S_i has $\circledcirc$-order type η then there exist subsets $P_i \subseteq S_i$ such that $\bigcup_{i \in d} P_i = X'$ for some $X \in \bar{\mathcal{K}}_3$ satisfying $(X, X{\upharpoonright}1) \subset\subset (Y_d, Y_d{\upharpoonright}1)$.

<u>proof</u>: Clear from Lemma 4.3 and its proof. □

<u>Theorem 4.14</u> For positive integers $d, k_0, k_1, \ldots, k_{d-1}$,

$$\begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} \longrightarrow \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix}^{k_0, k_1, \ldots, k_{d-1}}_{<\omega/\psi(k_0, k_1, \ldots, k_{d-1})}$$

and

$$\begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} \not\longrightarrow \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix}^{k_0, k_1, k_2, \ldots, k_{d-1}}_{<\omega/\psi(k_0, k_1, \ldots, k_{d-1})-1}$$

where $\psi(k_0, k_1, \ldots k_{d-1})$ is defined from the function ϕ (of Theorem 4.7) by

$$\psi(k_1, k_2, \ldots, k_{d-1}) = \left(\sum_{i \varepsilon d} (2k_i - 1)\right)! \prod_{i \varepsilon d} \frac{\phi(k_i)}{(2k_i - 1)!}.$$

Note that when $k_0 = k_1 = \ldots k_{d-1} = 1$, $\psi(k_0, k_1 \ldots, k_{d-1}) = d!$ which is Laver's result. When $d = 1$, the theorem reduces to Theorem 4.7 and indeed $\psi(k_0) = \phi(k_0)$.

<u>proof of 4.14</u> In analogy to the proof of Theorem 4.7 we use Lemma 4.12 to translate to the object $Y_d \varepsilon \mathcal{C}_3$. For each $A \varepsilon \mathcal{F}_3$ such that

(i) $(\forall i \varepsilon d)\left(|\{a \varepsilon A: \deg_A(a) = 0 \ \& \ \mathrm{root}(A) \prec_i a\}| = k_i\right)$

(ii) $\deg_A(\mathrm{root}(A)) = d$

(iii) $(\forall a, b \varepsilon A)\big((a \neq \mathrm{root}(A) \ \& \ a \prec b) \rightarrow (a \prec_0 b \text{ or } a \prec_1 b)\big)$

the translation induces a partition on $\begin{pmatrix} Y, & Y\rceil 1 \\ A, & A\rceil 1 \end{pmatrix}$.

Theorem 4.9 is then used repeatedly (once for each A which satisfies the above conditions) until finally $X \in \overline{\mathcal{K}}_3$ is produced such that $(X, X\restriction 1) \subset\subset (Y, Y\restriction 1)$ and the partition is constant on each $\binom{X, X\restriction 1}{A, A\restriction 1}$.

If $C \subseteq \bigcup_{i \in d} Q_i$ satisfies $(\forall i \in d)(|C \cap Q_i| = k_i)$ then clearly $\wedge$-closure $(C) \in \overline{\mathcal{F}}_3$ satisfies conditions (i) (ii) (iii) and its isomorphism type has been accounted for. For any A satisfying the conditions clearly $\binom{Y_d,\ Y_d\restriction 1}{A\ ,\ A\restriction 1} \neq 0$ and since any $X \in \mathcal{K}_3$ such that $(X, X\restriction 1) \subset\subset (Y_d, Y_d\restriction 1)$ is isomorphic to Y_d, we have $\binom{X,\ X\restriction 1}{A,\ A\restriction 1} \neq 0$. Lemma 4.13 translates this last fact back to the context of η order type sets and shows that whenever $P_i \subseteq Q_i$ are subsets of order type η $(i \in d)$ and $A \in \mathcal{F}_3$ satisfies the conditions (i) (ii) (iii) there will exist $C \subseteq \bigcup_{i \in d} P_i$ such that $(\forall i \in d)(|C \cap P_i| = k_i$ and the $\wedge$-closure of C is isomorphic to A).

The only thing left to do is count the objects A. For $i \in d$ let $A_i = \{a \in A: \mathrm{root}(A) \prec_i a\}$ and note $|\{a \in A_i: \deg_{A_i}(a) = 0\}| = k_i$, $|A_i| = 2k_i - 1$, $A_i \in \overline{\mathcal{K}}_3$ and there are $\phi(k_i)$ possibilities for the isomorphism type of A_i. $A = \bigcup_{i \in d} A_i \cup \{\mathrm{root}(A)\}$ is built from the A_i by interlacing the well orders $<^{A_i}$ and prefixing the root(A). The d-nomial coefficient

$$\binom{\sum_{i \in d}(2k_i - 1)}{(2k_0-1),\ (2k_1-1),\ \ldots,(2k_{d-1}-1)} = \frac{\left(\sum_{i \in d}(2k - 1)\right)!}{(2k_0-1)!(2k_1-1)!\ldots(2k_{d-1}-1)}$$

gives the number of ways this can be done. □

CHAPTER 5

A BASIC ULTRAFILTER CONSTRUCTION METHOD

Definition 5.1 A filter on a set, S, is a collection, $\mathcal{F}$, of subsets of S satisfying:

(1) $0 \notin \mathcal{F}$

(2) $X \in \mathcal{F}$ and $X \subseteq Y \subseteq S \rightarrow Y \in \mathcal{F}$

(3) $X, Y \in \mathcal{F} \rightarrow X \cap Y \in \mathcal{F}$

A filter $\mathcal{U} \subseteq \mathcal{P}(S)$ which satisfies the further condition

(4) $(\forall X \subseteq S)(X \in \mathcal{U}$ or $S - X \in \mathcal{U})$ is an ultrafilter.

To avoid triviality we also require the condition

(0) $X \subseteq S$ & $|X| < \omega \rightarrow S - X \in \mathcal{F}$, so for us, every filter extends the Frechet filter $\{X \subseteq S: |S - X| < \omega\}$. A filter or ultrafilter which satisfies (0) is usually called non-principal, but we will automatically assume the non-principal condition whenever the word 'filter' or 'ultrafilter' is used.

The definitions of an ideal and maximal ideal are dual, in the sense of the boolean algebra $\langle \mathcal{P}(S); \cap, \cup \rangle$, to the definitions of filter and ultrafilter. Again we include in our definition a non-triviality condition $(0')$, so for us, every ideal includes the finite subsets of S.

Definition 5.2 an ideal on S is a collection, $\mathcal{I}$, of subsets of S satisfying:

(0′) $X \subseteq S$ and $|X| < \omega \rightarrow X \in \mathcal{I}$

(1′) $S \notin \mathcal{I}$

(2′) $X \in \mathcal{I}$ and $Y \subseteq X \subseteq S \rightarrow Y \in \mathcal{I}$

(3′) $X, Y \in \mathcal{I} \rightarrow X \cup Y \in \mathcal{I}$

A maximal ideal satisfies the further condition

(4′) $(\forall X \subseteq S)(X \in \mathcal{I}$ or $S - X \in \mathcal{I})$

In addition to these familiar definitions we define a co-ideal on S as the compliment in $\mathcal{P}(S)$ of an ideal on S.

It is easily checked that a co-ideal, $\mathcal{H}$, on S is characterized by the conditions

(0″) $X \subseteq S$ & $|X| < \omega \rightarrow X \notin \mathcal{H}$

(1″) $S \in \mathcal{H}$

(2″) $X \in \mathcal{H}$ & $X \subseteq Y \subseteq S \rightarrow Y \in \mathcal{H}$

(3″) $X \cup Y \in \mathcal{H} \rightarrow X \in \mathcal{H}$ or $Y \in \mathcal{H}$

Definition 5.3 A collection $\mathcal{B} \subseteq \mathcal{P}(S)$ is a basis for a co-ideal $\mathcal{H}$ on S (or ultrafilter on S) iff $\mathcal{H} = \{X \subseteq S : (\exists B \in \mathcal{B})\ (B \subseteq X)\}$.

Given an ideal, $\mathcal{I}$, on S let $\mathcal{H} = \mathcal{P}(S) - \mathcal{I}$ be the corresponding co-ideal and $\mathcal{F} = \{S - X : X \in \mathcal{I}\}$ be the corresponding dual filter. For any ultrafilter $\mathcal{U}$ on S, we have,

$$\mathcal{F} \subseteq \mathcal{U} \rightarrow \mathcal{U} \subseteq \mathcal{G}.$$

Thus an ultrafilter on S can be viewed as a minimal co-ideal, a maximal filter, or the complement in $\mathcal{P}(S)$ of a maximal ideal.

As an illustration of a standard technique, we construct a Ramsey ultrafilter assuming the continuum hypothesis (CH).

Definition 5.4 An ultrafilter, $\mathcal{U}$, on a countable set S is Ramsey iff for every partition $\pi: [S]^2 \longrightarrow 2$ of the unordered pairs from S into 2 colors, there is an $X \in \mathcal{U}$ which is homogeneous with respect to π (i.e. π is constant on $[X]^2$).

This property of the collection of sets $\mathcal{U}$ is denoted $\mathcal{U} \longrightarrow [\mathcal{U}]^2_2$.

Definition 5.5 Given a collection, $\mathcal{S}$, of sets, the notation $\mathcal{S} \longrightarrow [\mathcal{S}]^n_r$ for cardinals n and r, means that for any partition $\pi: [S]^n \longrightarrow r$ where $S \in \mathcal{S}$ there is a set $T \subseteq S$ such that $T \in \mathcal{S}$ and $|\pi''[T]^n| < r$. We will usually be interested in cases where $n, r \in \omega$.

The notation $\mathcal{S} \not\longrightarrow [\mathcal{S}]^n_r$ means of course, the negation of $\mathcal{S} \longrightarrow [\mathcal{S}]^n_r$, so for some $S \in \mathcal{S}$ and some partition $\pi: [S]^n \longrightarrow r$ we have $|\pi\ [T]^n| = r$

whenever $T \subseteq S$ and $T \in \mathcal{S}$. In case this S satisfies $(\forall T \in \mathcal{S})(T \subseteq S)$, then a single "counterexample partition" $c: [S]^n \longrightarrow r$ satisfies $(\forall T \in \mathcal{S})(|c''[T]^n| = r)$, and for any non-empty $\mathcal{S}' \subseteq \mathcal{S}$ we have also $\mathcal{S}' \not\longrightarrow [\mathcal{S}]^n_r$.

We use the notation $\mathcal{S} \longmapsto [\mathcal{S}]^n_r$ to indicate $\mathcal{S} \rightarrow [\mathcal{S}]^n_{r+}$ (where r+ is the successor of r) and there exists $S \in \mathcal{S}$ such that $(\forall T \in \mathcal{S})(T \subseteq S)$ and there exists a partition $c: [S]^n \longrightarrow r$ such that $(\forall T \in \mathcal{S})(|c''[T]^n| = r)$.

<u>Example 5.6</u> Let Q be the rational numbers and put $\mathcal{S} = \{P \subseteq Q: P \text{ has order type } \eta\}$. Galvin's theorem says $\mathcal{S} \longmapsto [\mathcal{S}]^2_2$.

<u>Example 5.7</u> Given an ultrafilter $\mathcal{U}$ on ω, $\mathcal{U}$ is Ramsey iff $\mathcal{U} \longmapsto [\mathcal{U}]^2_1$.

To build a Ramsey ultrafilter on ω, we start with the co-ideal, $\mathcal{H}$, of infinite subsets of ω and note that $\mathcal{H} \longmapsto [\mathcal{H}]^2_1$ (this is just the infinite Ramsey theorem for a partition of pairs). For the purpose of later generalization we put $S = \omega$, $n = 2$ and $r = 1$. Using $2^{\aleph_0} = \aleph_1$ (CH) we can enumerate all partitions $c: [S]^n \longrightarrow r + 1$ as $\langle c_\gamma: \gamma \in \omega_1 \rangle$ and all subsets of S can be enumerated as $\langle S_\gamma: \gamma \in \omega_1 \rangle$. A basis, $\langle B_\gamma: \gamma \in \omega_1 \rangle$, for the ultrafilter $\mathcal{U}$ is constructed by induction of length ω_1.

Let $B_0 = S \in \mathcal{H}$. Having B_γ, choose $B_{\gamma+1}$ such that

1) $B_{\gamma+1} \in \mathcal{H}$

2) $B_{\gamma+1} \subseteq B_\gamma$

3) $|\pi_\gamma{}''[B_{\gamma+1}]^n| < r + 1$

4) $B_{\gamma+1} \subseteq S_\gamma$ or $B_{\gamma+1} \subseteq S - S_\gamma$

Since $\mathcal{H} \longmapsto [\mathcal{H}]^n_r$, there exists $B \in \mathcal{H}$ such that $B \subseteq B_\gamma$ and $|\pi_\gamma{}''[B]^n| < r + 1$. Now $B = (B \cap S_\gamma) \cup (B \cap (S - S_\gamma))$ so property $(3'')$ of a co-ideal can be used to put $B_{\gamma+1} = B \cap S_\gamma$ or $B_{\gamma+1} = B \cap (S - S_\gamma)$ so that the properties (1) through (4) above are satisfied.

To continue the inductive construction at a limit ordinal $\lambda \in \omega_1$ we assume that for $\mu\ \nu \in \lambda$ $(\mu < \nu \rightarrow B_\nu - B_\mu \notin \mathcal{H})$. Note that this condition is trivially maintained at the successor step defined above. We need to find $B_\lambda \in \mathcal{H}$ such that for $\mu < \nu$ $B_\lambda - B_\mu \notin \mathcal{H}$. The existence of such a B_λ follows from the countable completeness of the co-ideal $\mathcal{H}$, defined as follows:

<u>Definition 5.8</u> A co-ideal $\mathcal{H}$ on S is <u>countably complete</u> iff given any decreasing sequence $\langle C_n : n \in \omega \rangle$ such that $(\forall n \in \omega)(C_n \in \mathcal{H}\ \ \&\ \ C_n \supseteq C_{n+1})$ there exists $C \in \mathcal{H}$ which satisfies $(\forall n \in \omega)(C - C_n \notin \mathcal{H})$.

<u>Lemma 5.9</u> A co-ideal $\mathcal{H}$ on S is countably complete iff given any sequence $\langle B_\mu : \mu \in \lambda \rangle$ such that $\lambda \in \omega_1$, $B_\mu \in \mathcal{H}$, and $(\forall \mu,\nu \in \lambda)(\mu < \nu \rightarrow B_\nu - B_\mu \notin \mathcal{H})$ there exists $B \in \mathcal{H}$ such that $(\forall \mu \in \lambda)(B - B_\mu \notin \mathcal{H})$.

<u>proof</u>: $(\Leftarrow)$ Trivial. $(\Rightarrow)$ With $\langle B_\mu : \mu \in \lambda \rangle$ as above, let $f: \omega \hookrightarrow\!\!\!\rightarrow \lambda$ be a bijection and put $C_n = \bigcap_{m<n} B_{f(m)}$ so $S = C_0 \supseteq C_1 \supseteq C_2 \ldots$ Using the co-ideal properties of $\mathcal{H}$ and $(\mu < \nu \rightarrow B_\nu - B_\mu \notin \mathcal{H})$ it is easily seen that $(\forall n \in \omega)(C_n \in \mathcal{H})$. Since $\mathcal{H}$ is countably complete let $B \in \mathcal{H}$ satisfy $(\forall n \in \omega)(B - C_n \notin \mathcal{H})$. Since $(\forall \mu \in \lambda)(\exists n \in \omega)(C_n \subseteq B_\mu)$, clearly $(\forall \mu \in \lambda)(B - B_\mu \notin \mathcal{H})$. $\square$

<u>Lemma 5.10</u> The co-ideal $\mathcal{H}$ of infinite subsets of ω is countably complete.

<u>proof</u>: Clear. $\square$

In our construction of the ultrafilter base, $\langle B_\gamma : \gamma \in \omega_1 \rangle$, the combinatorial assumption $\mathcal{H} \longrightarrow [\mathcal{H}]^n_{r+1}$, is used at successor stages, and the countable completeness of $\mathcal{H}$ is used at limit stages. We must verify that $\mathcal{U} = \{X \subseteq S : (\exists \gamma \in \omega_1)(B_\gamma \subseteq X)\}$ is an ultrafilter on S and satisfies $\mathcal{U} \longmapsto [\mathcal{U}]^n_r$. Properties (1), (2), (4) in the definition of an ultrafilter (5.1) are clear by construction, and $\mathcal{U} \subseteq \mathcal{H}$ so the 'ultra property' (4) implies (0).

To verify condition (3), since $\mathcal{U}$ satisfies (4) and (0) it suffices to show $X,Y \in \mathcal{U} \to X \cap Y \neq 0$. But $X,Y \in \mathcal{U}$ means for some $\mu,\nu \in \omega_1$ $B_\mu \subseteq X$ and $B_\nu \subseteq Y$. Assume without loss of generality that $\mu < \nu$, so $B_\nu - B_\mu \notin \mathcal{H}$ and hence $B_\mu \cap B_\nu \in \mathcal{H}$ (using property (3″) of a coideal) so $X \cap Y \neq 0$. The construction guarantees $\mathcal{U} \longrightarrow [\mathcal{U}]^n_{r+1}$ and thus $\mathcal{U} \longmapsto [\mathcal{U}]^n_r$ follows from $\mathcal{H} \longmapsto [\mathcal{H}]^n_r$ and $\mathcal{U} \subseteq \mathcal{H}$.

By looking at the assumptions used for the construction of the Ramsey ultrafilter above, it is seen that the following theorem has actually been proven.

<u>Theorem 5.10</u> If $\mathcal{H}$ is a countably complete co-ideal on a countable set S and $\mathcal{H}$ satisfies $\mathcal{H} \longmapsto [\mathcal{H}]^n_r$ where $n,r \in \omega$ then assuming CH, there exists an ultrafilter $\mathcal{U}$ on S such that $\mathcal{U} \subseteq \mathcal{H}$ and $\mathcal{U} \longmapsto [\mathcal{U}]^n_r$.

Milliken's theorem and the partition theorems of Chapter 3 are good sources of co-ideals, $\mathcal{H}$, which satisfy $\mathcal{H} \longmapsto [\mathcal{H}]^n_r$ for some $n,r \in \omega$.

<u>Example 5.11</u> Let Y be the infinite binary tree, $Y = \bigcup_{n\in\omega} {}^n2$, structured as an object of $\mathcal{K}$. Let

$\mathcal{B} = \{X \subseteq Y : X \in \overline{\mathcal{K}}$ and $X \subset\subset Y\}$ and let

$\mathcal{H} = \{Z \subseteq Y : (\exists B \in \mathcal{B})\ (B \subseteq Z)\}$.

We claim that $\mathcal{H}$ is a co-ideal on Y and satisfies the

the partition property $\mathcal{H} \longmapsto [\mathcal{H}]^2_7$. Example 2.2 shows that $\mathbb{B} \longmapsto [\mathbb{B}]^2_7$ and thus $\mathcal{H} \longmapsto [\mathcal{H}]^2_7$. Clearly $\mathcal{H}$ satisfies properties (0″) (1″) and (2″) in the definition of a co-ideal (5.2). The condition (3″) that $S \cup T \in \mathcal{H} \rightarrow (S \in \mathcal{H}$ or $T \in \mathcal{H})$ can be regarded as the partition assertion $\mathcal{H} \longrightarrow [\mathcal{H}]^1_2$. Using Theorem 2.10, any partition $\pi : \binom{Y}{Y \restriction 1} \longrightarrow 2$ can be reduced by some $X \in \overline{X}$, and thus $\mathbb{B} \longrightarrow [\mathbb{B}]^1_2$ and $\mathcal{H} \longrightarrow [\mathcal{H}]^1_2$.

Hence $\mathcal{H}$ is a co-ideal on the countable set Y which satisfies $\mathcal{H} \longmapsto [\mathcal{H}]^2_7$. Unfortunately, if we consider the sequence $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ where $Y_n = \{y \in Y : (\forall m < n)(y(m) = 0)\}$, it is clear that $\mathcal{H}$ is not countably complete, and hence not directly usable for constructing an ultrafilter.

A countably complete co-ideal can be built from countably many copies of $\mathcal{H}$. The following definition generalizes the standard construction of a sum of ultrafilters.

<u>Definition 5.12</u> Given co-ideals $\mathcal{H}$ and $\mathcal{H}_n$ $(n \in \omega)$ on ω, define

$$\mathcal{H} \sum_{n \in \omega} \mathcal{H}_n = \{X \subseteq \omega \times \omega : \{n \mid \{m : \langle n,m \rangle \in X\} \in \mathcal{H}_n\} \in \mathcal{H}\}.$$

$\mathcal{H} \sum_{n \in \omega} \mathcal{H}_n$ is easily seen to be a co-ideal on $\omega \times \omega$.

<u>Lemma 5.13</u> $\mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$ (where $\mathcal{U}$, $\mathcal{U}_n$ are as above) is countably complete iff $\mathcal{U}$ is countably complete.

<u>Proof</u>: ($\Rightarrow$) Let $\langle C_n : n \in \omega\rangle$ be a descending sequence, $C_n \in \mathcal{U}$ and $C_n \supseteq C_{n+1}$. Then $\langle C_n \times \omega : n \in \omega\rangle$ is a descending sequence with $C_n \times \omega \in \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$ so there exists $\overline{C} \in \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$ such that $(\forall n \in \omega)\ \overline{C} - (C_n \times \omega) \notin \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$. Put $C = \{n \in \omega : \{m : (n,m) \in \overline{C}\} \in \mathcal{U}_n\}$ so $C \in \mathcal{U}$ and $(\forall n \in \omega)(C - C_n \notin \mathcal{U})$.

($\Leftarrow$) Let $\langle \overline{C}_n : n \in \omega\rangle$ be a descending sequence, $\overline{C}_n \in \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$, and put $C_n = \{p : \{q : (p,q) \in C_n\} \in \mathcal{U}_q\}$. Then $\langle C_n : n \in \omega\rangle$ is a descending sequence, $C_n \in \mathcal{U}$. In case $\bigcap_{n\in\omega} C_n \in \mathcal{U}$, let $\overline{C} = \{(p,q) : p \in \bigcap_{n\in\omega} C_n \ \&\ (p,q) \in \overline{C}_p\}$ so clearly $(\forall n \in \omega)(\overline{C} - C_n \notin \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n)$ and $\overline{C} \in \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$. Otherwise, $\bigcap_{n\in\omega} C_n \notin \mathcal{U}$ and there exists $C \in \mathcal{U}$ such that $(\forall n \in \omega)(C - C_n \notin \mathcal{U})$, and we can assume $C = \bigcup_{n\in\omega} C \cap (C_n - C_{n+1})$. Let $\overline{C} = \{(p,q) : \exists n(p \in C \cap (C_n - C_{n+1}) \ \&\ (p,q) \in \overline{C}_n)\}$. Again $\overline{C} \in \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$ and $(\forall n \in \omega)(\overline{C} - \overline{C}_n \notin \mathcal{U}\sum_{n\in\omega}\mathcal{U}_n)$. $\square$

<u>Definition 5.14</u> The product $\mathcal{U} \otimes \mathcal{U}_0$ where $\mathcal{U}$ and $\mathcal{U}_0$ are co-ideals on ω is defined as the sum $\mathcal{U}\sum_{n\in\omega}\mathcal{U}_n$ where $\mathcal{U}_0 = \mathcal{U}_1 = \mathcal{U}_2 = \cdots\cdots$.

Let $\mathcal{U}$ be the co-ideal of infinite subsets of ω and let $\mathcal{U}_0$ be the co-ideal of Example 5.11 which satisfies $\mathcal{U}_0 \longmapsto [\mathcal{U}_0]^2_7$ ($\mathcal{U}_0$ was previously referred to as $\mathcal{U}$). Then $\mathcal{U} \otimes \mathcal{U}_0$ is a countably complete co-ideal on $\omega \times Y$ from which an ultrafilter can be built. The problem now is that in passing to the product, $\mathcal{U} \times \mathcal{U}_0$, the combinatorial principal $\mathcal{U}_0 \longmapsto [\mathcal{U}_0]^2_7$ has been lost. In this simple case where we are only considering partitions of pairs it is not difficult to analyze all types of pairs which occur in $\omega \times Y$. In case a pair $\{x,y\} \subseteq \omega \times Y$ occurs in a single 'column', $\{x,y\} \subseteq \{n\} \times Y$ for some n, its type is determined by viewing $\{x,y\}$ as a subset of Y so the combinatorial properties of such pairs are governed by the partition property $\mathcal{U}_0 \longmapsto [\mathcal{U}_0]^2_7$. It can be shown that a pair $\{x,y$ which does not occur in a single column (so for some $n \neq m$ $x \in \{n\} \times Y$ and $y \in \{m\} \times Y$) is one of three distinct essential types so that $\mathcal{U} \otimes \mathcal{U}_0 \longmapsto [\mathcal{U} \otimes \mathcal{U}_0]^2_{10}$. This kind of analysis will become unwieldy as soon as we consider partitions of n-tuples where $n > 2$. For $n > 2$, an n-tuple which does not lie entirely within a single column could be spread across anywhere from 2 to n columns in many different ways.

Rather than develop a whole new notation and vocabulary for describing 'inter-column' types of n-tuple we pursue a different formalization which is analogous to the development of the notation and theorems for the category $\overline{\mathcal{C}}$. We will define the category $\bar{\mathcal{C}}_\omega = \bar{\mathcal{F}}_\omega \cup \bar{\mathcal{K}}_\omega$. An object of $\bar{\mathcal{K}}_\omega$ is intuitively a countable set of trees from $\bar{\mathcal{K}}$. Actually an object $Y \varepsilon \bar{\mathcal{K}}_\omega$ is a very highly structured countable set of trees in which the various types of finite substructure can be classified using isomorphisms to finite objects $A \varepsilon \mathcal{F}_\omega$. The details of our formalization of the category $\bar{\mathcal{C}}_\omega$ are somewhat arbitrary as usual. The goal is the partition theorem analogous to Theorem 2.10 but in the context of $\bar{\mathcal{C}}_\omega$.

CHAPTER 6

A PARTITION THEOREM FOR AN INFINITE FOREST

Recall from Chapter 1 the definition of $T = \bigcup_{n\in\omega} {}^n\omega$ with structure $\langle T;\prec,<,\ll,\prec_n,\wedge,\Delta\rangle_{n\in\omega} = \langle T;\sigma\rangle$.

Definition 6.1

$T_\omega^{(m)} = \{t \in T: |t| > m,\ t(0) = m \text{ and for } 0 < n \leqslant m \ \ t(n) = 0\}$ and put $T_\omega = \bigcup_{m\in\omega} T_\omega^{(m)}$.

The set $T_\omega^{(m)}$ is referred to as the m'th column of T_ω and $T_\omega(n) = \{t \in T: |t| = n + 1\}$ is referred to as the n'th level of T_ω. Note that $0 \in T - T_\omega$ and $T_\omega(0) = \{\langle 0\rangle\} \subseteq T(1)$. Put $T_\omega\restriction n = \bigcup_{p<n} T_\omega(p)$ and $T_\omega^{\restriction m} = \bigcup_{p<m} T_\omega^{(m)}$.

$T_\omega \subseteq T$ inherits the partial order $\prec$ from T and the $\prec$-minimal nodes of T_ω are called roots. Each column, $T_\omega^{(m)}$, has a unique root. In particular, $T_\omega^{(m)}$ has the node $f: m + 1 \longrightarrow m + 1$ defined by

$$f(i) = \begin{cases} m & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

as its root, and this root is the unique $\prec$-minimal node of T_ω which lies in $T_\omega(m)$. The root of $T_\omega^{(m)}$ is the $<$-maximal element of $T_\omega(m)$.

We note that T_ω is not closed under the operations of T, and hence we let T_ω inherit only the relational structure, $\prec, <, \ll, \prec_n$ $(n \in \omega)$, from T. New operation symbols $\curlywedge$ and $\blacktriangle$ are interpreted in T_ω instead of $\wedge$ and Δ. Given $x,y \in T_\omega$, if $x \Delta^T y \in T_\omega$ then $\blacktriangle$ (or $\blacktriangle^{T_\omega}$) is defined to agree with Δ^T, $x \blacktriangle y = x \Delta^T y$. The definition in the case $x \Delta^T y \notin T_\omega$ is made somewhat arbitrarily

$$x \blacktriangle y = \begin{cases} x \Delta y & \text{if } x \Delta y \in T_\omega \\ x & \text{if } x \Delta y \notin T_\omega \text{ and } x < y \\ y & \text{if } x \Delta y \notin T_\omega \text{ and } y < x \end{cases}$$

The same idea is used to define $\curlywedge$ for $x,y \in T_\omega$

$$x \curlywedge y = \begin{cases} x \wedge y & \text{if } x \wedge y \in T_\omega \\ x & \text{if } x \wedge y \notin T_\omega \text{ and } x < y \\ y & \text{if } x \wedge y \notin T_\omega \text{ and } y < x \end{cases}$$

Note $(\forall x,y \in T_\omega)(x \curlywedge y = x \wedge y)$ iff $(\exists m \in \omega)(x,y \in T_\omega^{(m)})$. If $x \blacktriangle y = x$ then in case $x \Delta y = x$ we conclude $x \preccurlyeq y$ and for some m; $x,y \in T_\omega^{(m)}$, but in case $x \Delta y \neq x$ (while $x \blacktriangle y = x$) then we conclude $x \in T_\omega^{(m)}$ and $y \in T_\omega^{(n)}$ where $m < n$.

The similarity type interpreted by T_ω will be denoted by σ_ω so $\sigma_\omega = \{\prec, <, <<, \prec_n, \curlywedge, \blacktriangle\}_{n\in\omega}$.

In analogy to the definition of $\mathcal{K}$ as a class of infinite canonical substructures of $\langle T;\sigma\rangle$, here we define the class $\mathcal{K}_\omega$ of infinite canonical substructures of $\langle T_\omega;\sigma_\omega\rangle$.

<u>Definition 6.2</u> Let the objects of $\mathcal{K}_\omega$ be the set of substructures of $\langle T_\omega;\sigma_\omega\rangle$ with universe $X \subseteq T_\omega$ which satisfies:

(i) $(\forall m \in \omega)$ $X \cap T_\omega^{(m)}$ is non-empty and closed under $\prec$-predecessors in T_ω.

(ii) Every $f \in X$ has a non-empty finite initial segment of immediate $\prec$-successors; $\widehat{f0}$, $\widehat{f1}$, $\widehat{f2}$, ... $\widehat{fn-1}$ for some $n > 0$. This n is called the degree of f in X, $\deg_X(f) = n$.

Given $X \in \mathcal{K}_\omega$ define

$X^{(m)} = X \cap T_\omega^{(m)}$ = the m'th column of X, and let

$X^{\upharpoonright m} = \bigcup_{p \in m} X^{(p)}$.

Put $X(n) = X \cap T_\omega(n)$ = the n'th level of X and let

$X \upharpoonright n = \bigcup_{p \in n} X(p)$. Note that $X \upharpoonright n \neq \{x \in X : |\{y \in X | \ y \prec x\}| < n\}$ (unless $n = 0$).

<u>Remark 6.3</u> We have chosen to use the symbols $\blacktriangle$ and $\bigwedge$ in the context of T_ω in order to emphasize the distinction from Δ and Λ. But it is clear that when restricted to a column of T_ω, $\blacktriangle$ and $\bigwedge$ agree with Δ and Λ respectively. More to the point, given $X \in \mathcal{K}_\omega$ we consider $X^{(m)} = X \cap T_\omega^{(m)}$ as an object of $\overline{\mathcal{K}}$, even though technically $X^{(m)}$ interprets the wrong similarity type, σ_ω. Given $X \in \mathcal{K}_\omega$, then $X = \bigcup_{m \in \omega} X^{(m)}$ and X can be viewed as an infinite collection of objects from $\overline{\mathcal{K}}$, but in addition the connection between the different objects $X^{(m)}$, $m \in \omega$, is highly structured.

For $m \in \omega$, $X(m) \cap X^{(m)}$ = {the root of $X^{(m)}$} = {the $<$-maximal element of $X(m)$}. For $n < m$, $X(n) \cap X^{(m)} = 0$. For $n \geq m$, $X(n) \cap X^{(m)} = X^{(m)}(n-m)$, where here we consider $X^{(m)} \in \overline{\mathcal{K}}$ and $X^{(m)}(n-m)$ denotes its level $n-m$.

The category $\mathcal{F}_\omega$ will be a set of canonical representatives for all isomorphism types of all possible finite substructures of objects $X \varepsilon \mathcal{K}_\omega$. In general, a finite substructure A of some $X \varepsilon \mathcal{K}_\omega$ may have several roots (= $\prec$ -minimal nodes) and in fact there may be several roots on the same level in A (where the level structure on A is inherited from X, rather than being defined from $\prec$). This fact makes the definition of a canonical representative of the isomorphism type of A somewhat awkward since in T_ω each level, $T_\omega(n)$, contains just one root of T_ω.

<u>Definition 6.4</u> Let the objects of $\mathcal{F}_\omega$ be the set of all substructures of $\langle T_\omega;\sigma_\omega \rangle$ with universe $A \subseteq T_\omega$ satisfying:

(i) A is non-empty and finite

(ii) $(\forall a \varepsilon A)(\forall t \varepsilon T_\omega - A)(t \prec a \rightarrow t \prec_0 a)$

(iii) If $A \cap T_\omega(n) \neq 0$ and for some $m < n$ A satisfies $\forall i\ (m \leqslant i < n \rightarrow A \cap T_\omega(i) = 0)$ then $|\{a \varepsilon A \cap T_\omega(n): a \text{ is a root of } A\}| > n - m$.

Given $A \varepsilon \mathcal{F}_\omega$ define

height(A) = $|\{n \varepsilon \omega: A \cap T_\omega(n) \neq 0\}|$ = the maximum cardinality $|S|$ where $S \subseteq A$ is linearly ordered by <<. For $n \varepsilon$ height(A) let $A(n) = \{a \varepsilon A: |\{b \varepsilon A: b << a\}| = n\}$ and let $A\lceil n = \bigcup_{m \varepsilon n} A(n)$.

As in the past the symbol X or A etc. will denote both an object and its universe and superscripts will only rarely be used to distinguish between a symbol in the similarity type, σ_ω , from its interpretation in a particular structure.

Definition 6.5 Put $\mathcal{C}_\omega = \mathcal{F}_\omega \cup \mathcal{K}_\omega$ and let $A \in \mathcal{F}_\omega$, $X \in \mathcal{K}_\omega$ and $Z \in \mathcal{C}_\omega$. As usual we define $\mathcal{C}_\omega(A,Z) = \{\langle A,\phi,Z\rangle \mid \phi: A \hookrightarrow Z$ is an isomorphic embedding of the σ_ω-structure A into the σ_ω-structure $Z\}$ and $\mathcal{C}_\omega(X,Z) = \{\langle X,\phi,Z\rangle \mid \phi: X \hookrightarrow Z$ is an isomorphic embedding of the σ_ω-structure X into the σ_ω-structure Z and $(\forall x \in X)(\deg_X(x) = \deg_Z(\phi(x)))\}$. $\mathcal{C}_\omega$ is a category (with composition defined as composition of maps) and $\mathcal{F}_\omega$ and $\mathcal{K}_\omega$ receive their morphisms as full subcategories of $\mathcal{C}_\omega$.

Lemma 6.6 Every isomorphism between objects of $\mathcal{C}_\omega$ is an identity map.

proof: Given an isomorphism $f: W \hookrightarrow\!\!\!\rightarrow Z$ between infinite objects $W,Z \in \mathcal{K}_\omega$, for each $n \in \omega$ f restricts to an isomorphism between $W\restriction n$ and $Z\restriction n$. But $W\restriction n$ and $Z\restriction n$ are finite canonical objects, so it suffices to consider only isomorphisms $f: A \hookrightarrow\!\!\!\rightarrow B$ where $A,B \in \mathcal{F}_\omega$. Since f preserves the well ordering $<$, there are no non-trivial automorphisms and we must show $A = B$.

Assume A and B consist entirely of roots and put $r = |A| = |B|$. If height(A) = 1 then $A \subseteq T_\omega(n)$ for some n, and we must have $n+1 \geqslant r$ and

$\forall i\ (0 \leqslant i < n \rightarrow A \cap T_\omega(i) = 0)$. Condition (iii) gives $|\{a \in A \cap T_\omega(n)\colon a \text{ is a root of } A\}| > n$ so $n+1 \geqslant r > n$, and $n = r-1$. Using (ii) and the closure of A under $\mathcal{A}$, this means $A = \{x \mid x \text{ is } <\text{-minimal in } T_\omega(r-1) \cap T_\omega^{(i)}$ for some i such that $0 \leqslant i \leqslant r-1\} = B$.

When height(A) > 1 the idea is just the same -- the conditions (iii) and (ii) are used as above to pin down precisely the set of roots in each level of A and B. When A and B do not consist entirely of roots the $\prec_n$-structure and condition (ii) are used to pin down these other nodes and show in general that A = B. □

<u>Definition 6.7</u> $\overline{\mathcal{F}}_\omega$ is the class of structures isomorphic (by a unique canonical isomorphism) to some object in $\widehat{\mathcal{F}}_\omega$.

$\overline{\mathcal{K}}_\omega$ is the class of structures isomorphic (by a unique canonical isomorphism) to some object in $\mathcal{K}_\omega$.

$\overline{\mathcal{C}}_\omega = \overline{\mathcal{F}}_\omega \cup \overline{\mathcal{K}}_\omega$ with morphisms defined by the correspondence to $\mathcal{C}_\omega = \widehat{\mathcal{F}}_\omega \cup \mathcal{K}_\omega$.

<u>Lemma 6.8</u> The finite subset $A \subseteq Z$ where $Z \in \overline{\mathcal{C}}_\omega$ is a sub-object, $A \in \overline{\mathcal{F}}_\omega$ and $A \subset\subset Z$ iff A is a substructure of Z.

<u>proof</u>: ($\Leftarrow$) The construction of the canonical isomorphism $\rho_A\colon A \hookrightarrow \langle T_\omega;\sigma_\omega\rangle$ from A into T_ω, whose image is a canonical object, $\rho_A(A) \in \widehat{\mathcal{F}}_\omega$ can be carried out by an inductive analysis of A very much like the proof of Lemma 6.6. □

<u>Lemma 6.9</u> Let $Y \in \overline{\mathcal{K}}_\omega$ and an infinite subset $Z \subseteq Y$ be given. Z is a sub-object of Y, $Z \in \overline{\mathcal{K}}_\omega$ and $Z \Subset Y$ iff

(i) Z is closed under Δ in Y

(ii) $(\forall z \in Z)(\deg_Z(z) = \deg_Y(z))$

(iii) $(\forall n \in \omega)\ (\exists!\ z \in Z(n))(z \text{ is a root of } Z)$.

Here $Z(n) = \{z \in Z: |\{w \in Z: w << z\}| = n\}$.

<u>proof</u>: routine. □

As in the case of objects of $\overline{\mathcal{K}}$, the structure of an object $\langle X;\prec,<,<<,\prec_n,\wedge,\Delta\rangle_{n\in\omega} \in \overline{\mathcal{K}}_\omega$ is definable from $\langle X;\prec,<\rangle$ and we could state conditions on a structure $\langle X;\prec,<\rangle$ so that the definitions yield an object $\langle X;\sigma_\omega\rangle$ (like Lemma 1.21). As in the case of an object of $\overline{\mathcal{F}}$, however, an object $\langle A;\sigma_\omega\rangle \in \overline{\mathcal{F}}_\omega$ is not definable from $\langle A;\prec,<\rangle$.

In the context of $\overline{\mathcal{C}}_\omega$, we are aiming to prove theorems analogous to the results of Milliken for the category $\overline{\mathcal{C}}$ (Chapter 2). For example we will obtain:

<u>Theorem</u> Given $A \in \overline{\mathcal{F}}_\omega$ and $Y \in \overline{\mathcal{K}}_\omega$ and a finite partition $\mathcal{c}: \binom{Y}{A} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_\omega$ such that X reduces $\mathcal{c}$.

Our proof of the theorem will proceed along lines parallel to the proof in the $\overline{\mathcal{C}}$ context, where it was necessary to state and prove more detailed theorems in order to construct inductive arguments.

Lemma 6.10 (= Lemma 2.4 for the category $\overline{\mathcal{C}}_\omega$)
Given $X, Y \in \overline{\mathcal{K}}_\omega$ and $n \in \omega$, if $(X, Y\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$ then $X\upharpoonright n+1$ is isomorphic to $Y\upharpoonright n+1$.

proof: By the degree preservation condition for sub-objects $(X, X\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$ where $X, Y \in \overline{\mathcal{K}}_\omega$, clearly $X\upharpoonright n = Y\upharpoonright n$. We define $\phi: Y\upharpoonright n+1 \hookrightarrow\!\!\!\rightarrow X\upharpoonright n+1$ for $y \in Y\upharpoonright n+1$ by

$$\phi(y) = \begin{cases} \text{the root of } X^{(n)} & \text{if } y = \text{the root of } Y^{(n)} \\ \text{the } \prec\text{-least } x \in X\upharpoonright n+1 \text{ s.t. } y \preccurlyeq x & \text{otherwise} \end{cases}$$

It is easily checked that ϕ is the required isomorphism $(Y\upharpoonright n+1, Y\upharpoonright n) \cong (X\upharpoonright n+1, X\upharpoonright n)$. $\square$

Lemma 6.11 (= Lemma 2.5 for the category $\overline{\mathcal{C}}_\omega$)
Given $Y \in \overline{\mathcal{K}}_\omega$ and $A \subset\subset Y$ where $A \in \overline{\mathcal{F}}_\omega$, let $A' = A\upharpoonright \text{height}(A)-1$, and suppose n satisfies $A' = A \cap Y\upharpoonright n \neq A \cap Y\upharpoonright n-1$. For any finite partition $\pi: \begin{pmatrix} Y, & A' \\ A, & A' \end{pmatrix} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_\omega$ such that $(X, X\upharpoonright n)$ reduces π.

proof: The proof here is considerably more complicated than the proof of the analogous Lemma 2.5, although like 2.5 the proof essentially rests on the Laver Pincus Theorem 1.26.

We can assume $Y \in \mathcal{K}_\omega$ and $A \in \hat{\mathcal{F}}_\omega$. As a first step toward fully reducing the partition $c: \binom{Y,\ A'}{A,\ A'} \longrightarrow r$, we find $Z \in \bar{\mathcal{K}}_\omega$ which satisfies the very weak reduction property expressed by the following:

$(Z, Z{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ and for any $B, C \in \binom{X,\ A'}{A,\ A'}$,

if $\{b \in B \mid (\exists x \in B)(b < x)\} = \{c \in C: (\exists x \in C)(c < x)\}$

then $c(B) = c(C)$.

Some notation will be useful, so for any $D \in \binom{Y,\ A'}{A,\ A'}$ let $\hat{D} = \{d \in D \mid (\exists x \in D)(d < x)\}$. The above property becomes $(Z, Z{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ and

† $\forall B, C \in \binom{Z,\ A'}{A,\ A'} (\hat{B} = \hat{C} \rightarrow c(B) = c(C))$. Having found such a $Z \in \bar{\mathcal{K}}_\omega$ satisfying †, there is a natural induced partition $\hat{c}: \binom{Z,\ A'}{\hat{A},\ A'} \longrightarrow r$ defined for $E \in \binom{Z,\ A'}{\hat{A},\ A'}$ by

$$\hat{c}(E) = \begin{cases} c(B) & \text{if there exists } B \in \binom{Z,\ A'}{A,\ A'} \text{ s.t. } \hat{B} = E \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Using the natural induced partition it is clear how to iterate the very weak reduction †, $|A - A'|$ times until finally reaching $X \in \bar{\mathcal{K}}_\omega$ which satisfies $(X, X{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ and c is constant on $\binom{X,\ A'}{A,\ A'}$.

The construction of $Z \in \bar{\mathcal{K}}_\omega$ satisfying † is split into cases depending essentially on whether there are any roots of A in $A - A'$.

<u>Case I</u> $A \subseteq Y^{\restriction n}$

By assumption $A' \cap Y(n-1) \neq 0$, so by Δ-closure of A, $(\forall a \in A - A')(\exists a' \in A \cap Y(n-1)(a' \prec a)$ and thus there are no roots of A in $A - A'$. Intuitively, we restrict attention to $Y\upharpoonright n$ and apply Lemma 2.5. Formally we must translate to the category $\overline{\mathcal{C}}$ by defining

$\overline{Y} = Y\upharpoonright n \cup \{t \in T: (\exists y \in Y(n-1))(t \preccurlyeq y)\}$

$\overline{A} = A \cup \{t \in T: (\exists y \in Y(n-1))(t \preccurlyeq y)\}$.

When $\overline{A}$ and $\overline{Y}$ are structured by inheritance from $\langle T;\sigma\rangle$, it is easily seen that $\overline{Y} \in \mathcal{K}$, $\overline{A} \in \overline{\mathcal{F}}$, $\overline{A} \subset\subset \overline{Y}$.

There is a natural 1-1 correspondence between

$$\begin{pmatrix} Y, & A' \\ A, & A' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{Y}, & \overline{A}' \\ \overline{A}, & \overline{A}' \end{pmatrix}$$

(where as usual $\overline{A}' = \overline{A}\upharpoonright \text{height}(\overline{A})-1$). Note that $\overline{Y}(0) = \{0\} = T(0) \neq T_\omega(0) \subseteq T(1)$ so $\overline{Y}(n) = Y(n-1)$ and $\overline{A}' = \overline{A}\upharpoonright n+1 = \overline{Y}\upharpoonright n+1 = \{t \in T: (\exists y \in Y(n-1))\ (t \preccurlyeq y)\}$.

The correspondence induced partition we denote

$$\overline{c}: \begin{pmatrix} \overline{Y}, & \overline{A}' \\ \overline{A}, & \overline{A}' \end{pmatrix} \longrightarrow r \quad \text{and it is defined for} \quad \overline{B} \in \begin{pmatrix} \overline{Y}, & \overline{A}' \\ \overline{A}, & \overline{A}' \end{pmatrix}$$

by $\overline{c}(\overline{B}) = c((\overline{B} - \overline{A}') \cup A')$. Note that $n+1$ satisfies $\overline{Y}\upharpoonright n+1 = \overline{A}' = \overline{A} \cap \overline{Y}\upharpoonright n+1 \neq \overline{A} \cap \overline{Y}\upharpoonright n = \overline{Y}\upharpoonright n$ so Lemma 2.5 gives $\overline{Z} \in \overline{\mathcal{K}}$ such that $(\overline{Z}, \overline{Z}\upharpoonright n+1)$ reduces $\overline{c}$.

The translation back to an object $Z \in \overline{\mathcal{K}}_\omega$ is carried out by defining the columns of Z, $Z \cap Y^{(p)}$ (so $Z \cap Y^{(p)} \subseteq T_\omega^{(p)}$ since Y is canonical), for each $p \in \omega$, and then letting $Z = \bigcup_{p\in\omega}(Z \cap Y^{(p)})$. Let L be the set of levels of Y used by $\overline{Z}$, $L = \{\ell \in \omega \mid Y(\ell) \cap \overline{Z} \neq 0\}$, so note $\overline{Z}\upharpoonright n+1 = \overline{Y}\upharpoonright n+1 \supseteq Y\upharpoonright n \rightarrow 0,1,2, \ldots, n-1 \in L$. For $p \in n$ let $Z \cap Y^{(p)} = \overline{Z} \cap Y^{(p)}$. Additional columns are now added to complete Z to an object of $\overline{\mathcal{K}}_\omega$ (being careful to have exactly one root on each level of Z). For $p \notin L$

let $Z \cap Y^{(p)}$ be empty. For $p \varepsilon L - n$, note $Y^{(p)}$ has its root at level p in Y, and $Y^{(p)} \varepsilon \bar{\mathcal{K}}$. We let $Z \cap Y^{(p)}$ be any subobject, $Z \cap Y^{(p)} \subset\subset Y^{(p)}$ (in the sense of $\bar{\mathcal{K}}$) which uses levels $L - p$, i.e., $\{\ell \varepsilon \omega | (Z \cap Y^{(p)}) \cap Y(\ell) \neq 0\} = L - p$. Since $Z \cap Y^{(p)}$ "uses level p" we note for future reference that whenever $Z \cap Y^{(p)}$ is defined to be non-empty (i.e. $p \varepsilon L$) then $\{\text{root } (Z \quad Y^{(p)})\} = \{\text{root } (Y^{(p)})\} = T_{\omega}^{(p)} \cap T_{\omega}(p)$ and hence every root of Z is a root of Y.

It is easily checked that $Z \varepsilon \bar{\mathcal{K}}_{\omega}$ when structured by inheritance from Y or T_{ω}, and $(Z, Z\restriction n) \subset\subset (Y, Y\restriction n)$. Following the 1-1 correspondence between $\begin{pmatrix} \bar{Z}, & \bar{A}' \\ \bar{A}, & \bar{A}' \end{pmatrix}$ and $\begin{pmatrix} Z, & A' \\ A, & A' \end{pmatrix}$

we conclude $(Z, Z\restriction n)$ reduces $\mathcal{C}$, and thus certainly Z satisfies the much weaker assertion $\dagger$.

<u>Case II</u> $A \not\subseteq Y^{\restriction n}$

In this case there is at least one root of A in $A - A'$. $Z \varepsilon \bar{\mathcal{K}}_{\omega}$ which satisfies $\dagger$ is produced by a sequence of two constructions, IIa and IIb. First we define for any $X \varepsilon \bar{\mathcal{K}}_{\omega}$ where $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$ and for any $D \varepsilon \begin{pmatrix} X, & A' \\ A, & A' \end{pmatrix}$, "the projection of D to minimal level in X" by $\pi_X(B) = A' \cup \{x \varepsilon X(\ell) | (\exists d \varepsilon D)(x \preccurlyeq d)\}$ where $\ell \varepsilon \omega$ is minimal such that $(\forall d \varepsilon D - A')(\exists x \varepsilon X(\ell))(x \preccurlyeq d)$. Since we are assuming $A \not\subseteq Y^{\restriction n}$ and $(X, X\restriction n) \subset\subset (Y, Y\restriction n)$, clearly $\ell \geqslant n$ and $\pi_X(D)$ is isomorphic to D and A. The minimal level, ℓ, is determined by which column of X contains the <-greatest element of D (= the element of $D - \hat{D}$).

If $D - \hat{D} \subseteq X^{(p)}$ then $\ell = p$ and
$\pi_X(D) - \pi_X(D) = \{\text{the root of } X^{(p)}\}$.

Construction IIa will yield $\tilde{Y} \in \overline{\mathcal{K}}_\omega$ which satisfies $(\tilde{Y}, \tilde{Y}\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$ and for any $B, C \in \begin{pmatrix} \tilde{Y}, & A' \\ A, & A' \end{pmatrix}$

†† $$\pi_{\tilde{Y}}(B) = \pi_{\tilde{Y}}(C) \rightarrow \mathcal{c}(B) = \mathcal{c}(C).$$

Construction IIb will yield $Z \in \overline{\mathcal{K}}_\omega$ such that $(Z, Z\upharpoonright n) \subset\subset (Y, Y\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$ and Z satisfies †, but the method used to construct Z will depend on whether $A' = \hat{A}$. In case IIb(i), $\hat{A} = A'$ (i.e. $|A - A'| = 1$ and we are still assuming $A \nsubseteq Y^{\upharpoonright n}$), a simple construction inside $\tilde{Y}$ gives $Z \in \overline{\mathcal{K}}_\omega$ which satisfies $(Z, Z\upharpoonright n)$ $(\tilde{Y}, \tilde{Y}\upharpoonright n)$ and $\mathcal{c}$ is constant on $\begin{pmatrix} Z, & A' \\ A, & A' \end{pmatrix}$ (so certainly Z satisfies †).
In case IIb(ii), $\hat{A} \neq A'$ (and $A \nsubseteq Y^{\upharpoonright n}$), the construction of Z is more complicated. First we define for any $X \in \overline{\mathcal{K}}_\omega$ where $(X, X\upharpoonright n) \subset\subset (\tilde{Y}, \tilde{Y}\upharpoonright n)$ and any $D \in \begin{pmatrix} X, & A' \\ A, & A' \end{pmatrix}$ the "completion of $\hat{D}$ in co-minimal level of X" by
$\pi^*_X(\hat{D}) = X\upharpoonright\ell \cup \{x \in X(\ell) \mid (\exists d \in \hat{D})(x \preccurlyeq d)\}$ where $\ell \in \omega$ is minimal such that $(\forall d \in \hat{D})(\exists x \in X(\ell-1)(x \preccurlyeq d))$ and $\ell \geqslant n$.
The co-minimal level, ℓ, defined above, can also be determined as $\ell = \max\{n, p+1\}$ where $X^{(p)}$ contains the <-maximal element of $\hat{D}$. Note that $(\pi^*_X(\hat{D}))' = X\upharpoonright\ell$ (where the "prime" notation has the usual meaning of restriction to height$(\pi^*_X(\hat{D})) - 1$) and $\pi^*_X(\hat{D}) \subseteq X^{\upharpoonright\ell}$ where $\ell \geqslant n$.

The construction of Z in case IIb(ii) will yield $Z \in \overline{\mathcal{K}}_\omega$ such that $(Z, Z\upharpoonright n) \subset\subset (\tilde{Y}, \tilde{Y}\upharpoonright n) \subset\subset (Y, Y\upharpoonright n)$ and for any $B,C \in \begin{pmatrix} Z, & A' \\ A, & A' \end{pmatrix}$, $\pi^*_Z(\hat{B}) = \pi^*_Z(\hat{C}) \rightarrow \mathcal{C}(B) = \mathcal{C}(C)$.

Then for any $B,C \in \begin{pmatrix} Z, & A' \\ A, & A' \end{pmatrix}$ such that $\hat{B} = \hat{C}$ we have $\pi^*_Z(\hat{B}) = \pi^*_Z(\hat{C})$ so $\mathcal{C}(B) = \mathcal{C}(C)$ and Z satisfies $\dagger$.

<u>Construction IIa</u> A sequence of objects $\overset{t}{Y} \in \overline{\mathcal{K}}_\omega$ is constructed for $t \in \omega - n$ beginning with $\overset{n}{Y} = Y$ and satisfying

(i) $\forall\, B,C \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$

if $\pi_{\overset{t}{Y}}(B) = \pi_{\overset{t}{Y}}(C) \subseteq \overset{t}{Y}\,t$

then $\mathcal{C}(B) = \mathcal{C}(C)$

(ii) $(\overset{t+1}{Y}, \overset{t+1}{Y}\!\upharpoonright\! t) \subset\subset (\overset{t}{Y}, \overset{t}{Y}\!\upharpoonright\! t)$

We claim that condition (i) is satisfied vacuously by $\overset{n}{Y} = Y$. Since we are in case II where $A - A'$ includes a root of A, $B - A'$ includes a root of B. Since $B \cap Y(n-1) = A' \cap Y(n-1) \neq 0$ and B is closed under Δ, this root of B ocurrs in some $Y^{(p)}$ for $p \geqslant n$. But then $\pi_Y(B) \cap Y_{(p)} \neq 0$ so $\pi_Y(B) \nsubseteq Y\!\upharpoonright\! n$.

Given $\overset{t}{Y}$ satisfying (i), $\overset{t+1}{Y}$ is obtained by a modification of the technique used in case I of this proof. By following the canonical isomorphism $\rho_{\overset{t}{Y}} : \overset{t}{Y} \hookrightarrow \langle T;\sigma_\omega \rangle$ we treat $\overset{t}{Y}$ as a canonical object $\overset{t}{Y} \in \mathcal{K}_\omega$. In analogy to case I, we will restrict attention to $\overset{t}{Y}\!\upharpoonright\! t+1$ (<u>not</u> $\overset{t}{Y}\!\upharpoonright\! t$ which would be the exact analogy to Case I -- a crucial difference). Define

$\overline{Y} = \overset{t}{Y}\upharpoonleft t+1 \cup \{x \in T \mid \exists y \in \overset{t}{Y}(t)\ (x \prec y)\}$ and for fixed

$D \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$ which satisfies $\pi_{\overset{t}{Y}}(D) \subseteq \overset{t}{Y}\upharpoonleft t+1$ & $\pi_{\overset{t}{Y}}(D) \subseteq \overset{t}{Y}\upharpoonleft t$

(i.e. $D - \hat{D} \subseteq \overset{t}{Y}(t)$) define

$\overline{D} = D \cup \{x \in T \mid \exists y \in \overset{t}{Y}(t)\ (x \prec y)\}$. [Note for the first few values of $t = n, n+1, n+2, \ldots,$ there may not exist any such $D \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$, in which case the combinatorial arguments used to construct $\overset{t+1}{Y}$ are vacuous and we can simply put $\overset{t+1}{Y} = \overset{t}{Y}$.]

When $\overline{D}$ and $\overline{Y}$ are structured by inheritance from $\langle T; \sigma \rangle$, the following facts are easily checked: $\overline{Y} \in \mathcal{K}$, $\overline{D} \in \overline{\mathcal{F}}$, $\overline{D} \subset\subset \overline{Y}$, $\overline{Y}(t+1) = \overset{t}{Y}(t)$,

$\overline{Y}(t) = \overset{t}{Y}(t-1) \cup \{$the immediate $\prec$-predecessor in $\langle T;\sigma\rangle$ of root$(\overset{t}{Y}(t))\}$, and $\overline{D}' = \overline{D}\upharpoonleft t+1 = \overline{Y}\upharpoonleft t+1 =$

$\{x \in T \mid \exists y \in \overset{t}{Y}(t)\ (x < y)\}$. Note that the $<$-maximal node, y, of $\overline{Y}\upharpoonleft t+1$ is not a node of $\overline{Y}$ (nor of T_ω) and $\deg_{\overline{Y}}(y) = 1$ so $(\forall z \in \overline{Y})(y \prec z \rightarrow y \prec_0 z)$.

There is a natural 1-1 correspondence between

$\{B \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix} \mid \pi_{\overset{t}{Y}}(B) = \pi_{\overset{t}{Y}}(D)\}$ and $\begin{pmatrix} \overline{Y}, & \overline{D}' \\ \overline{D}, & \overline{D}' \end{pmatrix}$. Given

$B \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$ such that $\pi_{\overset{t}{Y}}(B) = \pi_{\overset{t}{Y}}(D)$, then $B \subseteq \overset{t}{Y}\upharpoonleft t+1$

and $B \cap \overset{t}{Y}(t)$ is the $<$-maximal node, b, of B. Letting $y \in \bar{Y}$ be the $<$-maximal node of $\bar{Y}\upharpoonright t+1$, we have $y \prec_0 b$. Put $\bar{B} = B \cup \bar{Y}\upharpoonright t+1$ and now it is easily seen that $\bar{B} \in \begin{pmatrix} \bar{Y}, & \bar{D}' \\ \bar{D}, & \bar{D}' \end{pmatrix}$. Going the other way, given $\bar{B} \in \begin{pmatrix} \bar{Y}, & \bar{D}' \\ \bar{D}, & \bar{D}' \end{pmatrix}$ put $B = (\bar{B} - \bar{D}') \cup A'$ so $B \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$ and $\pi_{\overset{t}{Y}}(B) = \pi_{\overset{t}{Y}}(D)$.

The induced partition $\bar{c}\colon \begin{pmatrix} \bar{Y}, & \bar{D}' \\ \bar{D}, & \bar{D}' \end{pmatrix} \longrightarrow r$ is defined for $B \in \begin{pmatrix} \bar{Y}, & \bar{D}' \\ \bar{D}, & \bar{D}' \end{pmatrix}$ by $\bar{c}(\bar{B}) = c((\bar{B} - \bar{D}) \cup A')$. Now $\bar{Y}\upharpoonright t+1 = \bar{D}' = \bar{D} \cap \bar{Y}\upharpoonright t+1 \neq \bar{D} \cap \bar{Y}\upharpoonright t = \bar{Y}\upharpoonright t$ so Lemma 2.5 gives $\bar{X} \in \bar{\mathcal{K}}$ such that $(\bar{X}, \bar{X}\upharpoonright t+1) \subset\!\subset (\bar{Y}, \bar{Y}\upharpoonright t+1)$ and $\bar{c}$ is constant on $\begin{pmatrix} \bar{Y}, & \bar{D}' \\ \bar{D}, & \bar{D}' \end{pmatrix}$. In other words, for any $B, C \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$, if $B, C \subseteq \bar{X}$ and $\pi_{\overset{t}{Y}}(B) = \pi_{\overset{t}{Y}}(C) = \pi_{\overset{t}{Y}}(D)$, then $c(B) = c(C)$.

By repeating the argument which produces $\bar{X}$, for each of the (at most finitely many) distinct projections, $\pi_{\overset{t}{Y}}(D)$, where $D \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$ satisfies $(\pi_{\overset{t}{Y}}(D) \subseteq \overset{t}{Y}\upharpoonright t+1 \;\&\; \pi_{\overset{t}{Y}}(D) \not\subseteq \overset{t}{Y}\upharpoonright t)$, we construct a finite chain of sub-objects of $\bar{Y}$. The last of these sub-objects we again call $\bar{X}$ (in order to avoid extra notation) and $\bar{X} \in \bar{\mathcal{K}}$ satisfies:

$(\bar{X}, \bar{X}\upharpoonright t+1) \subset\!\subset (\bar{Y}, \bar{Y}\upharpoonright t+1)$ and for any $B, C \in \begin{pmatrix} \overset{t}{Y}, & A' \\ A, & A' \end{pmatrix}$,

if $B, C \subseteq \bar{X}$ and

$((\pi_{\underset{Y}{t}}(B) = \pi_{\underset{Y}{t}}(C) \subseteq \overset{t}{Y}\restriction t+1)$ & $(\pi_{\underset{Y}{t}}(B) = \pi_{\underset{Y}{t}}(C) \not\subseteq \overset{t}{Y}\restriction t))$, then $\not{c}(B) = \not{c}(C)$. Now the inductive assumption (i) on $\overset{t}{Y}$ says $(\forall B,C \in \begin{pmatrix}\overset{t}{Y}, & A' \\ A, & A'\end{pmatrix})(\pi_{\underset{Y}{t}}(B) = \pi_{\underset{Y}{t}}(C) \subseteq \overset{t}{Y}\restriction t \rightarrow \not{c}(B) = \not{c}(C))$.

Hence the property satisfied by $\overline{X} \subset\subset \overline{Y}$ can be strengthened to read:

$$(\forall B,C \in \begin{pmatrix}\overset{t}{Y}, & A' \\ A, & A'\end{pmatrix})(B,C \subseteq \overline{X} \;\&\; \pi_{\underset{Y}{t}}(B) = \pi_{\underset{Y}{t}}(B) \subseteq \overset{t}{Y}\restriction t+1 \rightarrow \not{c}(B) = \not{c}(C)).$$

The translation from $\overline{X} \in \overline{\mathcal{K}}$ back to an object $\overset{t+1}{Y} \in \mathcal{K}_\omega$ is carried out by a method similar to case I. Let $L = \{\ell \in \omega \mid \overset{t}{Y}(\ell) \cap \overline{X} \neq 0\}$ so note $\overline{X}\restriction t+1 = \overline{Y}\restriction t+1 \supseteq Y\restriction t \rightarrow 0,1,2 \ldots t-1 \in L$. The important columns of $\overset{t+1}{Y}$ which control the partition property (i) for $\overset{t+1}{Y}$ are defined for $0 \leqslant p < t+1$ by $\overset{t+1}{Y} \cap \overset{t}{Y}{}^{(p)} = \overline{X} \cap \overset{t}{Y}{}^{(p)}$. For $p \in L - t$, $\overset{t+1}{Y} \cap \overset{t}{Y}{}^{(p+1)}$ is defined arbitrarily as any sub-object of $\overset{t}{Y}{}^{(p+1)}$ (in the sense of $\overline{\mathcal{K}}$) which uses levels $L - (p+1)$. For $0 \leqslant p \notin L$, the intersection of $\overset{t+1}{Y}$ with column $\overset{t}{Y}{}^{(p+1)}$ is defined to be empty. Note that $\mathrm{root}(\overset{t+1}{Y} \cap \overset{t}{Y}{}^{(t)}) \in \overset{t}{Y}(\ell)$ where $\ell \in L$ is minimal such that $\ell \geqslant t$. Typically $t \notin L$, so $\mathrm{root}(\overset{t+1}{Y} \cap \overset{t}{Y}^{(t)})$ is not a root of $\overset{t}{Y}$. More generally for $p \in L$, $\mathrm{root}(\overset{t+1}{Y} \cap \overset{t}{Y}^{(p+1)}) \in \overset{t}{Y}(\ell)$ where $\ell \in L$ is minimal such that $\ell \geqslant p+1$, and since this is minimal, $\overset{t+1}{Y}$ is closed under Δ in $\overset{t}{Y}$. Also $\{\mathrm{root}(\overset{t+1}{Y} \cap \overset{t}{Y}^{(0)})\} = \overset{t}{Y}(0)$, so for every $\ell \in L$, there exists exactly one root of $\overset{t+1}{Y}$ in $\overset{t}{Y}(\ell)$, and it is apparent that $\overset{t+1}{Y} \in \overline{\mathcal{K}}_\omega$ and $(\overset{t+1}{Y}, \overset{t+1}{Y}\restriction t) \subset\subset (\overset{t}{Y}, \overset{t}{Y}\restriction t)$. Since $\overset{t+1}{Y}\restriction t+1 \subseteq \overset{t}{Y}\restriction t+1$ we note that $\forall\, B,C \in \begin{pmatrix} \overset{t+1}{Y} ; & A' \\ A ; & A' \end{pmatrix}$

$$(\pi_{\substack{t+1\\Y}}(B) = \pi_{\substack{t+1\\Y}}(C) \subseteq \overset{t+1}{Y}\restriction t+1) \rightarrow (\pi_{\substack{t\\Y}}(B) = \pi_{\substack{t\\Y}}(C) \subseteq \overset{t}{Y}\restriction t+1).$$

So the construction of $\overset{t+1}{Y}$ from $\overline{X}$ gives

$$(\pi_{\substack{t+1\\Y}}(B) = \pi_{\substack{t+1\\Y}}(C) \subseteq \overset{t+1}{Y}\restriction t+1) \rightarrow \mathcal{c}(B) = \mathcal{c}(C) .$$

Having defined the sub-object chain $\overset{t}{Y}$ for $t \in \omega-n$, put $\tilde{Y} = \bigcap_{t\in\omega-n} \overset{t}{Y}$, and since $\overset{t+1}{Y}\restriction t = \overset{t}{Y}\restriction t$, note $\tilde{Y} = \bigcup_{t\in\omega-n} \overset{t}{Y}\restriction t \in \overline{\mathcal{K}}_\omega$ and $(\tilde{Y}, \tilde{Y}\restriction n) \subset\subset (Y, Y\restriction n)$. Given $B,C \in \begin{pmatrix} \tilde{Y}, & A' \\ A, & A' \end{pmatrix}$ with $\pi_{\tilde{Y}}(B) = \pi_{\tilde{Y}}(C)$, for some $t \in \omega-n$, $\pi_{\substack{t+1\\Y}}(B) = \pi_{\substack{t+1\\Y}}(C) \subseteq \overset{t+1}{Y}\restriction t+1$. Hence $\tilde{Y}$ satisfies the conclusion of construction IIa,

$$\dagger\dagger \quad (\forall\ B,C \in \begin{pmatrix}\tilde{Y}, & A' \\ A, & A'\end{pmatrix})\ (\pi_{\tilde{Y}}(B) = \pi_{\tilde{Y}}(C) \rightarrow \kappa(B) = \kappa(C)\ .$$

<u>Construction IIb(i)</u> $\hat{A} = A'$ and $A \not\subseteq Y{\restriction}n$. We have $(\tilde{Y}, \tilde{Y}{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ satisfying $\dagger\dagger$, and now inside $\tilde{Y}$ we will find $Z \in \overline{\mathcal{K}}_\omega$ such that $(Z, Z{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ and κ is constant on $\begin{pmatrix}Z, & A' \\ A, & A'\end{pmatrix}$. Since $\hat{A} = A'$ means $|A - A'| = 1$, for any $B,C \in \begin{pmatrix}\tilde{Y}, & A' \\ A, & A'\end{pmatrix}$

$\pi_{\tilde{Y}}(B) = \pi_{\tilde{Y}}(C) \leftrightarrow B - \hat{B}$ and $C - \hat{C}$ are in the same column of $\tilde{Y}$.

For each $c \in r$ let

$L_c = \{p \in \omega |\ B - \hat{B} \in \tilde{Y}^{(p)} \rightarrow \kappa(B) = c\}$ so $n \subseteq L_c$ (vacously) and $L_0 \cup L_1 \cup \ldots \cup L_{r-1} = \omega$. One of these sets (call it L) must be infinite. Define the columns of $Z \in \overline{\mathcal{K}}_\omega$ by letting $Z \cap \tilde{Y}^{(p)}$ be empty if $p \notin L$, and for $p \in L$ define $Z \cap \tilde{Y}^{(p)}$ as an arbitrary sub-object of $\tilde{Y}^{(p)}$ (in the sense of $\overline{\mathcal{K}}$) which uses levels $L - p$ (i.e.,

$\{\ell \in \omega |\ (Z \cap \tilde{Y}^{(p)}) \cap \tilde{Y}(\ell) \neq 0\} = L - p$). Since $n \subseteq L$, $(Z, Z{\restriction}n) \subset\subset (\tilde{Y}, \tilde{Y}{\restriction}n) \subset\subset (Y, Y{\restriction}n)$ where $Z \in \overline{\mathcal{K}}_\omega$ and clearly $(Z,Z{\restriction}n)$ reduces κ.

<u>Construction IIb(ii)</u>, $\hat{A} \neq A'$ and $A \subseteq Y{\restriction}n$. We want $Z \in \overline{\mathcal{K}}_\omega$ such that $(Z, Z{\restriction}n) \subset\subset (\tilde{Y}, \tilde{Y}{\restriction}n)$ and

$(\forall B,C \in \begin{pmatrix}Z, & A' \\ A, & A'\end{pmatrix})(\pi^*_Z(\hat{B}) = \pi^*_Z(\hat{C}) \rightarrow \kappa(B) = \kappa(C))$.

Let $\mathcal{S} = \{D \in \begin{pmatrix}\tilde{Y}, & A' \\ A, & A'\end{pmatrix} : D - \hat{D}$ is a root of $\tilde{Y}\}$

$$= \{D \in \begin{pmatrix}\tilde{Y}, & A' \\ A, & A'\end{pmatrix} : D = \pi_{\tilde{Y}}(D)\}$$

Note that any $X \subset\subset \widetilde{Y}$, $X \in \overline{\mathcal{K}}_\omega$ such that $(\forall x \in X)(x \text{ is a root of } X \rightarrow x \text{ is a root of } \widetilde{Y})$ will satisfy $(\forall D \subseteq X)(\pi_X(D) = \pi_{\widetilde{Y}}(D))$ and

$\binom{X}{\mathcal{S}} = \{D \in \begin{pmatrix} X, & A' \\ A, & A' \end{pmatrix} : D - \widehat{D} \text{ is root of } X\}$. A sequence of objects $\overset{t}{Z} \in \overline{\mathcal{K}}_\omega$ is constructed for $t \in \omega - n$ beginning with $\overset{n}{Z} = \widetilde{Y}$ and satisfying:

(i) For any $B, C \in \binom{\overset{t}{Z}}{\mathcal{S}}$

$$(\pi^*_{\overset{t}{Z}}(\widehat{B}) = \pi^*_{\overset{t}{Z}}(\widehat{C}) \subseteq \overset{t}{Z}\restriction t) \rightarrow c(B) = c(C)$$

(ii) $(\overset{t+1}{Z}, \overset{t+1}{Z}\restriction t) \subset\subset (\overset{t}{Z}, \overset{t}{Z}\restriction t)$

(iii) every root of $\overset{t+1}{Z}$ is a root of $\overset{t}{Z}$.

Recall for any $(X, X\restriction n) \subset\subset (\widetilde{Y}, Y\restriction n)$ and any $D \in \begin{pmatrix} X, & A' \\ A, & A' \end{pmatrix}$ the definition of $\pi^*_X(\widehat{D})$ requires $X\restriction n \subseteq \pi^*_X(\widehat{D})$. Hence for $\overset{n}{Z} = \widetilde{Y}$ there is no $D \in \binom{\overset{n}{Z}}{\mathcal{S}}$ such that $\pi^*_{\overset{n}{Z}}(\widehat{D}) = \overset{n}{Z}\restriction n$ and condition (i) is satisfied vacuously by $\overset{n}{Z}$.

Given $\overset{t}{Z}$ satisfying (i) the construction of $\overset{t+1}{Z}$ which satisfies (ii) and (iii) is essentially the construction of case I, but this time it will not be necessary to translate all the way back to the category $\overline{\mathcal{C}}$.

Given a fixed $D \in \binom{\overset{t}{Z}}{\mathcal{S}}$ such that $\left(\pi^{*}_{\overset{t}{Z}}(\hat{D})\right)' = \overset{t}{Z}\upharpoonright t$

put $\bar{D} = \pi^{*}_{\overset{t}{Z}}(\hat{D}) \in \bar{\mathcal{F}}_{\omega}$ (not $\bar{\mathcal{F}}$) and to simplify notation

let $\bar{X} = \overset{t}{Z} \in \bar{\mathcal{K}}_{\omega}$ (not $\bar{\mathcal{K}}$). Note that

$\bar{D}' = \bar{X}\upharpoonright t = \bar{D} \cap \bar{X}\upharpoonright t \neq \bar{D} \cap \bar{X}\upharpoonright t-1$ where $\bar{D} \subset\subset \bar{X}$ (in the sense of $\bar{\mathcal{C}}_{\omega}$), and $\bar{D} \subseteq \bar{X}^{\upharpoonright t}$ (since $(\forall d \in \pi^{*}_{\overset{t}{Z}}(\hat{D}))(\exists z \in \overset{t}{Z}(t-1))(z \prec d)$. For any finite partition

$\bar{c}: \binom{\bar{X},\ \bar{X}\upharpoonright t}{\bar{D},\ \bar{X}\upharpoonright t} \longrightarrow r$, case I of this proof gives $\bar{W} \in \bar{\mathcal{K}}_{\omega}$

($\bar{W}$ corresponds to the Z mentioned at the end of case I) such that

$(\bar{W}, \bar{W}\upharpoonright t)$ reduces $\bar{c}$ and

$(\forall w \in \bar{W})$(w is a root of $\bar{W}$ $\rightarrow$ w is a root of $\bar{X}$ $\rightarrow$ w is a root of $\tilde{Y}$)

The partition $\bar{c}: \binom{\bar{X},\ \bar{X}\upharpoonright t}{\bar{D},\ \bar{X}\upharpoonright t} \longrightarrow r$ which we have in mind

is defined from c be following the natural 1-1 correspondence between $\binom{\bar{X},\ \bar{X}\upharpoonright t}{\bar{D},\ \bar{X}\upharpoonright t}$ and $\{B \in \binom{\bar{X}}{\mathcal{S}} \mid \pi^{*}_{\bar{X}}(\hat{B}) = \bar{D}\}$.

Given $E \in \binom{\bar{X},\ \bar{X}\upharpoonright t}{\bar{D},\ \bar{X}\upharpoonright t}$ let $\bar{c}(E) = c(B)$ where $B \in \binom{\bar{X}}{\mathcal{S}}$

is the unique element such that $E = \hat{B} \cup \bar{X}\upharpoonright t$ (i.e.,

$B = (E - \bar{X}\upharpoonright t) \cup A' \cup$ {the <-maximal element of $\bar{X}(\ell)$ where

$E = E \cap \bar{X}\upharpoonright \ell+1 \neq E \cap \bar{X}\upharpoonright \ell$ }). By following the correspondence which defined $\bar{c}$ from c, the fact that $(\bar{W}, \bar{W}\upharpoonright t)$ reduces $\bar{c}$ implies

$\forall B,C \in \binom{\bar{X}}{\mathcal{S}}$

if $\pi^{*}_{\bar{X}}(\hat{B}) = \pi^{*}_{\bar{X}}(\hat{C}) = \bar{D}$ and $B,C \subseteq \bar{W}$ then $c(B) = c(C)$.

Recall now that $\overline{X}$ was duplicate notation for $\overset{t}{Z}$ and

$\overline{D} = \pi^*_{\underset{Z}{t}}(\hat{D})$ where $D \in \binom{\overset{t}{Z}}{\mathcal{S}}$ satisfying $\left(\pi^*_{\underset{Z}{t}}(\hat{D})\right)' = \overset{t}{Z}\upharpoonright t$

was fixed. By repeating the argument which produces $\overline{W} \subset\subset \overset{t}{Z}$ for each of the (at most finitely many) distinct 'cominimal completions', $\overline{D} = \pi^*_{\underset{Z}{t}}(\hat{D})$, where $D \in \binom{\overset{t}{Z}}{\mathcal{S}}$ satisfies

$\left(\pi^*_{\underset{Z}{t}}(\hat{D})\right)' = \overset{t}{Z}\upharpoonright t$, we construct a finite chain of sub-objects of $\overset{t}{Z}$. The last of these sub-objects, we call $\overset{t+1}{Z}$ and $\overset{t+1}{Z} \in \overline{\mathcal{K}}_\omega$ satisfies:

(ii) $(\overset{t+1}{Z}, \overset{t+1}{Z}\upharpoonright t) \subset\subset (\overset{t}{Z}, \overset{t}{Z}\upharpoonright t)$

(iii) $(\forall z \in \overset{t+1}{Z})(z \text{ is a root of } \overset{t+1}{Z} \rightarrow z \text{ is a root of } \overset{t}{Z})$

$\forall \quad B,C \in \binom{\overset{t}{Z}}{\mathcal{S}}$

* if $\left(\pi^*_{\underset{Z}{t}}(\hat{B})\right)' = \left(\pi^*_{\underset{Z}{t}}(\hat{C})\right)' = \overset{t}{Z}\upharpoonright t$ and $B,C \subseteq \overset{t+1}{Z}$

then $rc(B) = rc(C)$.

Given $B,C \in \binom{\overset{t+1}{Z}}{\mathcal{S}}$ such that

$\pi^*_{\underset{Z}{t+1}}(\hat{B}) = \pi^*_{\underset{Z}{t+1}}(\hat{C}) \subseteq \overset{t+1}{Z}\upharpoonright t+1$ then using properties (ii) and

(iii) for $\overset{t+1}{Z}$, we have $\pi^*_{\underset{Z}{t}}(\hat{B}) = \pi^*_{\underset{Z}{t}}(\hat{C}) \subseteq \overset{t}{Z}\upharpoonright t+1$. Either

$\left(\pi^{*}_{\overset{t}{Z}}(\hat{B})\right)' = \left(\pi^{*}_{\overset{t}{Z}}(\hat{C})\right)' = \overset{t}{Z}\restriction t$, in which case $\ast$ gives $\mathscr{c}(B) = \mathscr{c}(C)$, or $\pi^{*}_{\overset{t}{Z}}(\hat{B}) = \pi^{*}_{\overset{t}{Z}}(\hat{C}) \leq \overset{t}{Z}\restriction t$, in which case property (i) for $\overset{t}{Z}$ gives $\mathscr{c}(B) = \mathscr{c}(C)$. Hence $\overset{t+1}{Z}$ satisfies (i) and the inductive construction of the $\overset{t}{Z}$, $t \in \omega - n$, can continue.

Having defined the sub-object chain $\overset{t}{Z}$ for all $t \in \omega - n$, put $Z = \bigcap_{t\in\omega-n} \overset{t}{Z} = \bigcup_{t\in\omega-n} \overset{t}{Z}\restriction t$ and note that $Z \in \overline{\mathcal{K}}_{\omega}$ satisfies

$(Z,\ Z\restriction n) \subset\subset (\tilde{Y},\ \tilde{Y}\restriction n)$ and

$(\forall z \in Z)(z$ is a root of $Z \to z$ is a root of $\tilde{Y})$.

Let $D,E \in \binom{Z,\ A'}{A,\ A'}$ satisfy $\pi^{*}_{Z}(\hat{D}) = \pi^{*}_{Z}(\hat{E})$. We want to show $\mathscr{c}(D) = \mathscr{c}(E)$. Put $B = \pi_{Z}(D)$ and $C = \pi_{Z}(E)$. Note that since roots of Z are roots of $\tilde{Y}$ in fact $B = \pi_{\tilde{Y}}(D)$ and $C = \pi_{\tilde{Y}}(E)$. We have $B,C \in \binom{Z}{\Delta}$ and $\pi^{*}_{Z}(\hat{B}) = \pi^{*}_{Z}(\hat{D}) = \pi^{*}_{Z}(\hat{E}) = \pi^{*}_{Z}(\hat{C})$. Hence for some $t \in \omega - n$ $B,C \in \binom{\overset{t}{Z}}{\Delta}$ and $\pi^{*}_{\overset{t}{Z}}(\hat{B}) = \pi^{*}_{\overset{t}{Z}}(\hat{C}) \leq \overset{t}{Z}\restriction t$ so $\mathscr{c}(B) = \mathscr{c}(C)$. Since $B,C,D,E \in \binom{\tilde{Y},\ A'}{A,\ A'}$ satisfy $\pi_{\tilde{Y}}(D) = \pi_{\tilde{Y}}(B)$ and $\pi_{\tilde{Y}}(C) = \pi_{\tilde{Y}}(E)$, property †† gives finally

$$\mathscr{c}(D) = \mathscr{c}(B) = \mathscr{c}(C) = \mathscr{c}(E). \qquad \square$$

Lemmas 6.12, 6.13, 6.14 are proved by exactly the same arguments which prove their counterparts back in the category $\bar{\mathcal{C}}$.

<u>Lemma 6.12</u> (= Lemma 2.7 for the category $\bar{\mathcal{C}}_\omega$)

Let $\mathcal{A} \subseteq \mathcal{F}_\omega$, $Y \in \bar{\mathcal{K}}_\omega$ and $n \in \omega$ satisfy

(i) $\forall A \in \binom{Y}{\mathcal{A}}$ $(A' = A \cap Y\upharpoonright n \neq A \cap Y\upharpoonright n-1)$

(ii) $\forall A,B \in \binom{Y}{\mathcal{A}}$ $(A' = B' \rightarrow A$ is isomorphic to $B)$

For any finite partition $c:\binom{Y}{\mathcal{A}} \rightarrow r$ there exists $X \in \bar{\mathcal{K}}_\omega$ such that $(X, X\upharpoonright n)$ weakly reduces c.

<u>Lemma 6.13</u> (= Lemma 2.8 for the category $\bar{\mathcal{C}}_\omega$)

Let $\mathcal{A} \subseteq \bar{\mathcal{F}}_\omega$, $Y \in \bar{\mathcal{K}}_\omega$ and $n \in \omega$ satisfy

(i) $\forall A \in \binom{Y}{\mathcal{A}}$ $(A' \supsetneq A \cap Y\upharpoonright n-1)$

(ii) $\forall A,B \in \binom{Y}{\mathcal{A}}$ $(A' = B' \rightarrow A$ is isomorphic to $B)$

For any finite partition $c:\binom{Y}{\mathcal{A}} \rightarrow r$ there exists $X \in \bar{\mathcal{K}}_\omega$ such that $(X, X\upharpoonright n)$ weakly reduces c.

<u>Theorem 6.14</u> (= Theorem 2.9 for the category $\bar{\mathcal{C}}_\omega$)

Let $\mathcal{A} \subseteq \bar{\mathcal{F}}_\omega$, $Y \in \bar{\mathcal{K}}_\omega$, $n \in \omega$, $C \in \bar{\mathcal{F}}_\omega$ satisfy:

(i) $C \subset\subset Y\upharpoonright n$, $C \not\subseteq Y\upharpoonright n-1$

(ii) $\forall A \in \binom{Y}{\mathcal{A}}$ $(A \cap Y\upharpoonright n = C)$

(iii) $\forall A,B \in \binom{Y}{\mathcal{A}}$ (height(A) = height(B))

(iv) $(\forall m \in \omega)(\forall A,B \in \binom{Y}{\mathcal{A}})$

$(A\upharpoonright m = B\upharpoonright m \rightarrow A\upharpoonright m+1$ is isomorphic to $B\upharpoonright m+1)$.

For any finite partition $c:\binom{Y}{\mathcal{A}} \rightarrow r$ there exists $X \in \bar{\mathcal{K}}_\omega$ such that $(X, X\upharpoonright n)$ reduces c.

<u>Theorem 6.15</u> (= Theorem 2.10 for the category $\bar{\mathcal{C}}_\omega$)

For any $Y \in \bar{\mathcal{K}}_\omega$ and any $A \in \bar{\mathcal{F}}_\omega$ and any finite partition $c:\binom{Y}{A} \rightarrow r$ there exists $X \in \bar{\mathcal{K}}_\omega$ such that X reduces c.

proof: By using isomorphic copies of Y and A (if necessary), we can assume $A \subset\subset Y$ where Y is a substructure of $\langle T_\omega;\sigma_\omega\rangle$ and satisfies $\{\ell \in \omega: Y \cap T_\omega(\ell) \neq 0\} = \omega - 1$. Let $\bar{Y}^{(0)} \in \bar{\mathcal{K}}$ satisfy $\{\ell \in \omega: \bar{Y}^{(0)} \cap T_\omega(\ell) \neq 0\} = \omega$ and $Y^{(0)} \subset\subset T_\omega^{(0)})$. Put $\bar{Y} = \bar{Y}^{(0)} \cup Y$. Let $\bar{A} = A \cup T_\omega(0)$. When structured by inheritance from $\langle T_\omega;\sigma_\omega\rangle$ clearly $\bar{A} \in \bar{\mathcal{F}}_\omega$, $\bar{Y} \in \bar{\mathcal{K}}_\omega$ and there is an obvious 1-1 correspondence between

$$\binom{\bar{Y},\ \bar{Y}\restriction 1}{\bar{A},\ \bar{Y}\restriction 1} \quad \text{and} \quad \binom{Y}{A}.$$

Given $\bar{B} \in \binom{\bar{Y},\ \bar{Y}\restriction 1}{\bar{A},\ \bar{Y}\restriction 1}$ let $\bar{c}(\bar{B}) = c(\bar{B} - \bar{Y}\restriction 1)$. Let $C = \bar{Y}\restriction 1$, $n = 1$, and $\mathcal{A} = \binom{\bar{Y},\ \bar{Y}\restriction 1}{\bar{A},\ \bar{Y}\restriction 1}$. The conditions of Theorem 6.14 are satisfied giving $\bar{X} \in \bar{\mathcal{K}}_\omega$ such that $(\bar{X}, \bar{X}\restriction 1)$ reduces $\bar{c}$. Let $X = \bar{X} - \bar{X}^{(0)}$. Clearly $X \in \bar{\mathcal{K}}_\omega$ reduces c. □

Theorem 6.16 (= Theorem 2.11 for the category $\bar{\mathcal{C}}_\omega$).

Given $Y \in \bar{\mathcal{K}}_\omega$ and finite $n > 1$, let $\mathcal{A} = \{X\restriction n: (X, X\restriction 1) \subset\subset (Y, Y\restriction 1) \text{ where } X \in \bar{\mathcal{K}}_\omega\}$

For any finite partition $c: \binom{Y}{\mathcal{A}} \longrightarrow r$ there exists $X \in \bar{\mathcal{K}}_\omega$ such that $(X, X\restriction 1)$ reduces c.

proof: This follows from Theorem 6.14 just as Theorem 2.11 follows from Theorem 2.9. □

CHAPTER 7

ULTRAFILTERS CONSTRUCTED ON TREES

Let $Y \in \overline{\mathcal{K}}_\omega$ be a fixed object and let $\mathcal{U} = \{S \subseteq Y: \exists X \in \overline{\mathcal{K}}_\omega, X \subset\subset Y \text{ and } X \subseteq S\}$. Since any partition $c: \binom{X}{X \upharpoonright 1} \longrightarrow 2$ can be reduced by some $Z \in \mathcal{K}_\omega$, $\mathcal{U}$ is a co-ideal on Y. If every $X \subset\subset Y$ where $X \in \overline{\mathcal{K}}_\omega$ has nodes of arbitrarily large degree, then every $S \in \mathcal{U}$ will clearly have infinitely many distinct types of pair, triple, etc., so we have the partition property $\mathcal{U} \longmapsto [\mathcal{U}]^n_\omega$ for $2 < n < \omega$. If, on the other hand, there exists $X \subset\subset Y$ such that $\{\deg_X(x) | x \in X\}$ is bounded, then by partitioning the nodes according to degree and applying Theorem 6.15, there exists $Z \subset\subset X \subset\subset Y$, $Z \in \overline{\mathcal{K}}_\omega$ such that for some fixed $d \in \omega$ $(\forall z \in Z)(\deg_Z(z) = d)$. For such a uniform Z put $\deg(Z) = d$. Since any sub-object $W \subset\subset Z$ where $W \in \overline{\mathcal{K}}_\omega$ also satisfies $\deg(W) = d$, W is in fact isomorphic to Z. Hence the partition properties of $\mathcal{U}_Z = \{S \subseteq Z: \exists W \in \overline{\mathcal{K}}_\omega, W \subset\subset Z \text{ and } W \subseteq S\}$ can be determined by analyzing the finite substructures of the uniform object Z.

<u>Example 7.1</u> If $\deg(Z) = 2$, a pair $\{x,y\} \in [Z]^2$, is classified according to the isomorphism type (in the sense of $\mathcal{F}_\omega$) of the A-closure of $\{x,y\}$. Since each column, $Z^{(p)}$, of Z is isomorphic to $\bigcup_{n \in \omega} {}^n 2$ as an object of $\overline{\mathcal{K}}$, the type of any intra-column pair, $\{x,y\} \subseteq Z^{(p)}$ for some $p \in \omega$, is one of the 7 possibilities enumerated in

Example 2.2. The possibilities for the type of any inter-column pair, $\{x,y\}$ such that $x \varepsilon Z^{(p)}$ & $y \varepsilon Z^{(q)}$ where $p \neq q$, are enumerated as follows:

• • $= \{\langle 0,0\rangle, \langle 1,0\rangle\}$

• . $= \{\langle 0\rangle, \langle 1,0\rangle\}$

• / $= \Delta\text{-closure } \{\langle 0,0\rangle, \langle 1,0,0\rangle\}$
$\quad = \{\langle 0,0\rangle, \langle 1,0,0\rangle, \langle 1,0\rangle\}$

• \ $= \Delta\text{-closure } \{\langle 0,0\rangle, \langle 1,0,1\rangle\}$
$\quad = \{\langle 0,0\rangle, \langle 1,0,1\rangle, \langle 1,0\rangle\}$

/ • $= \Delta\text{-closure } \{\langle 0,0,0\rangle, \langle 1,0\rangle\}$
$\quad = \{\langle 0,0,0\rangle, \langle 1,0\rangle, \langle 0,0\rangle\}$

\ • $= \Delta\text{-closure } \{\langle 0,0,1\rangle, \langle 1,0\rangle\}$
$\quad = \{\langle 0,0,1\rangle, \langle 1,0\rangle, \langle 0,0\rangle\}$

Hence $\mathcal{D}_Z \longmapsto [\mathcal{D}_Z]^2_{13}$ (where $Z \varepsilon \mathcal{K}_\omega$, $\deg(Z) = 2$). More generally the finite substructures of Z which are closures of an n-tuple can be counted to give a finite number, $\phi(n)$, which satisfies $\mathcal{D}_Z \longmapsto [\mathcal{D}_Z]^n_{\phi(n)}$. Assuming CH there exists an ultrafilter $\mathcal{U}$ on Z such that $\mathcal{U} \subseteq \mathcal{D}_Z$ and $\mathcal{U}$ satisfies these same partition properties but we are not quite ready to apply Theorem 5.10.

<u>Lemma 7.2</u> Given $Y \varepsilon \bar{\mathcal{K}}_\omega$ such that $(\forall y \varepsilon Y)(\deg_Y(y) \geqslant 2)$ let $\mathcal{D} = \{S \subseteq Y : (\exists X \varepsilon \bar{\mathcal{K}}_\omega)(X \subset\subset Y \text{ and } X \subseteq Z)\}$. $\mathcal{D}$ is <u>not</u> countably complete.

<u>proof</u>: Let $\omega = \bigcup_{n\in\omega} L_n$ be a partition of ω into infinitely many disjoint infinite sets. For each $m \in \omega$ let $R_m \subseteq \bigcup_{\ell\in L_m} Y(\ell)$ satisfy $\forall p\ (|R_m \cap Y^{(p)}| = 1)$. By an easy inductive construction (using $(\forall y \in Y)\deg_Y(y) \geqslant 2$) we can also assume $(\forall r, r' \in \bigcup_{m\in\omega} R_m)\ (r \not< r'\ \&\ r' \not< r)$.

Given $r \in \bigcup_{m\in\omega} R_m$, suppose $r \in Y^{(p)} \cap R_m$ and let $X_r \in \overline{\mathcal{K}}$ satisfy $X_r \subset\subset Y^{(p)}$ and $(X_r \cap Y(\ell) \neq 0 \leftrightarrow \ell \in L_m\ \&\ \exists y \in Y(\ell)\ (r \leqslant y))$. Put $B_n = \bigcup\{X_r \mid r \in \bigcup_{m\geqslant n} R_m\}$. We claim $B_n \in \mathcal{U}$ for all $n \in \omega$. We know $R_n \subseteq \bigcup_{\ell\in L_n} Y(\ell)$ satisfies $(\forall p \in \omega)(|R_n \cap Y^{(p)}| = 1)$, so beginning with $R_n \cap Y^{(0)} = \{r_0\}$, a sequence $\langle r_i ; i\in\omega\rangle \subseteq R_n$ can be chosen such that $(\forall i \in \omega)(r_i << r_{i+1}\ \&\ r_i \wedge r_{i+1} = r_i)$. Let $\overline{L}_n = \{\ell \in \omega\colon \text{for some } i,\ r_i \in Y(\ell)\} \subseteq L_n$, and for each $i \in \omega$ let $\overline{X}_{r_i} \in \overline{\mathcal{K}}$ satisfy

$$\overline{X}_{r_i} \subset\subset X_{r_i}\ \&\ \left(\overline{X}_{r_i} \cap Y(\ell) \neq 0 \leftrightarrow (\ell \in \overline{L}_n\ \&\ \exists y \in Y(\ell)(r_i \leqslant y))\right)$$

Put $X = \bigcup_{i\in\omega} \overline{X}_{r_i}$, so clearly $X \in \overline{\mathcal{K}}_\omega$ and $X \subset\subset Y$ (see Lemma 6.11). Since $X \subseteq \bigcup_{r\in R_n} X_r \subseteq B_n$ we conclude $B_n \in \mathcal{U}$.

Clearly $B_0 \supsetneqq B_1 \supsetneqq B_2 \supsetneqq \cdots$. Suppose $B \in \mathcal{U}$ satisfies $\forall n\ (B - B_n) \notin \mathcal{U}$, and we may as well assume $B \subseteq B_0$ and $B = X$ for some $X \in \overline{\mathcal{K}}_\omega$, $X \subset\subset Y$. Let $x = \mathrm{root}(X^{(0)})$ and let $L = \{\ell \in \omega \mid X \cap Y(\ell) \neq 0\}$. Now $X^{(0)} \subseteq B_0$, so for some $r \in \bigcup_{m\in\omega} R_m$, $x \in X_r$. Since distinct $r, r' \in \bigcup_{m\in\omega} R_m$ are $\prec$-incomparable, $x \in X_r$ & $X^{(0)} \subseteq B_0 \rightarrow X^{(0)} \subseteq X_r$. For some $m \in \omega$, $r \in R_m$ and $\{\ell \in \omega : X_r \cap Y(\ell) \neq 0\} \subseteq L_m$ so $L \subseteq L_m$. But $(\forall \ell \in L_m)(B_{m+1} \cap Y(\ell) = 0)$ and hence $X \cap B_{m+1} = 0$. contrary to $X - B_{m+1} \notin \mathfrak{U}$. □

This counterexample to countable completeness of is based on the fact that there exist $W, X \in \overline{\mathcal{K}}_\omega$ such that $W \subset\subset Y$, $X \subset\subset Y$, and $\forall n\ (W \cap Y(n) \neq 0 \rightarrow X \cap Y(n) = 0)$. Given a fixed Ramsey ultrafilter $\mathbb{R}$, the smaller co-ideal $\{S \subseteq Y: (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y$ & $\{\ell \in \omega |\ X \cap Y(\ell) \neq 0\} \in \mathbb{R}$ & $X \subseteq S)\}$ avoids this counterexample, while Theorem 6.15 can be strengthened to prove partition properties for this smaller co-ideal.

The method used to strengthen Theorem 6.15 is applicable quite generally and is based on some descriptive set theory and a theorem due to Mathias [6].

<u>Definition 7.3</u> A class of infinite sets $\mathcal{A} \subseteq [\omega]^\omega$ is <u>Ramsey</u> iff $\exists S \in [\omega]^\omega$ such that $([S]^\omega \subseteq \mathcal{A}$ or $[S]^\omega \subseteq [\omega]^\omega - \mathcal{A})$.

A topology is induced on $[\omega]^\omega$ by identifying each $S \in [\omega]^\omega$ with its characteristic function, $\chi_S \in {}^\omega 2$, where the topology on ${}^\omega 2$ is the Tychanoff product topology and 2 is the two point discrete space. Silver proved [9] that every analytic set $\mathcal{A} \subseteq [\omega]^{\omega}$ is Ramsey. Mathias strengthened this result.

<u>Theorem 7.4</u> (Mathias [6])

Let $\mathcal{R}$ be a Ramsey ultrafilter on ω. If $\mathcal{A} \subseteq [\omega]^{\omega}$ is analytic then
$(\exists S \in \mathcal{R})([S]^{\omega} \subseteq \mathcal{A}$ or $[S]^{\omega} \subseteq [\omega]^{\omega} - \mathcal{A})$.

<u>Corollary 7.5</u> If the analytic set $\mathcal{A}$ is in addition dense in the partial ordering $\langle [\omega]^{\omega}, \subseteq \rangle$, (i.e., $(\forall T \in [\omega]^{\omega})(\exists S \in \mathcal{A})(S \subseteq T)$), then in fact $(\exists S \in \mathcal{R})$ such that $[S]^{\omega} \subseteq \mathcal{A}$.

<u>proof</u>: Since $S \in \mathcal{R}$ is infinite and $\mathcal{A}$ is dense in $[\omega]^{\omega}$, $[S]^{\omega} \subseteq [\omega]^{\omega} - \mathcal{A}$ is impossible. □

<u>Theorem 7.6</u> Let $\mathcal{R}$ be a Ramsy ultrafilter on ω. For any $Y \in \overline{\mathcal{K}}_{\omega}$, $A \in \overline{\mathcal{F}}_{\omega}$, and any finite partition $\mathcal{C}: \binom{Y}{A} \longrightarrow r$ there exists $X \in \overline{\mathcal{K}}_{\omega}$ such that X reduces $\mathcal{C}$ and $\{\ell \in \omega | X \cap Y(\ell) \neq 0\} \in \mathcal{R}$.

<u>proof</u>: Let $\mathcal{A} = \{L \subseteq \omega | (\exists X \in \overline{\mathcal{K}}_{\omega})(X \text{ reduces } \mathcal{C} \ \& \ L = \{\ell \in \omega | X \cap Y(\ell) \neq 0\})\}$. Using Theorem 6.15, is $\mathcal{A}$ easily seen to be dense in the partial ordering $\langle [\omega]^{\omega}, \subseteq \rangle$. The theorem will follow from Corollary 7.5, once $\mathcal{A}$ is shown to be analytic. For this we use the basic fact from

descriptive set theory which identifies the analytic subsets of $[\omega]^\omega$ with the Σ^1_1 definable subsets. There are many ways to state this fact precisely, and here we will give the minimum of detail for a formalization most suited for use in this proof.

Let L^2_N be an applied second order language for the natural numbers N (or ω), which has first order variables n,s,t,u, etc. ranging over ω and second order variables S,T,U,etc. ranging over $[\omega]^\omega$. A set $S \varepsilon [\omega]^\omega$ can be regarded (by some fixed scheme) as the code for a countable sequence, $\langle (S)_n : n \varepsilon \omega \rangle$, of sets $(S)_n \varepsilon [\omega]^\omega$. The language L^2_N includes a parameter for the decoding function (which takes a set variable, S, and a number variable, n, as arguments to give the set $(S)_n$). A number $s \varepsilon \omega$ can be regarded (by some fixed scheme) as the code for a finite sequence of numbers $\langle (s)_n : n \varepsilon \ell h(s) \rangle$ where $\ell h(s) \varepsilon \omega$ is the length of the finite sequence. There are parameters in L^2_N for the length and the decoding functions (which take number variables as arguments).

The enlarged language $\underset{\sim}{L}^2_N$ has in addition, constant parameters for all $S \varepsilon [\omega]^\omega$.

We need a formula of $\underset{\sim}{L}^2_N$ defining $\mathcal{A} \subseteq [\omega]^\omega$, which begins with a second order existential quantifier and all other quantifiers are first order. Here is our previous definition of $\mathcal{A}$ stretched out somewhat:

$L \varepsilon \mathcal{A}$ iff $\exists S \varepsilon [\omega]^\omega$ such that S is the code for some

X which satisfies,

(i) X is a substructure of Y

(ii) $(\forall x \in X)(\deg_X(x) = \deg_Y(x))$

(iii) $(\forall B,C \in \overline{\mathcal{F}}_\omega)(B,C \in \binom{X}{A} \to c(B) = c(C))$

(iv) $(\forall \ell \in \omega)(\ell \in L \leftrightarrow \exists x \in X \cap Y(\ell))$

Let $f: Y \hookrightarrow\!\!\!\to \omega$ be a fixed bijection. Put $\dot{Y} = f(Y) = \omega$, and induce the structure $\langle \dot{Y}; \prec^{\dot{Y}}, <^{\dot{Y}}, <<^{\dot{Y}}, \prec_n^{\dot{Y}}, A^{\dot{Y}}, \Delta^{\dot{Y}} \rangle_{n\in\omega}$ so that f is an isomorphism. Although $\prec^{\dot{Y}} \subseteq \omega\times\omega$, $\Delta^{\dot{Y}} \subseteq \omega\times\omega\times\omega$, etc., by coding finite sequences as singletons we can regard these as subsets of ω. In this sense, $\langle Y;\sigma_\omega \rangle$ has been coded as a countable sequence of subsets of ω. This sequence in turn is coded by a single set which we refer to with the constant parameter $\underset{\sim}{Y}$. As an example, we illustrate a formalization of the assertion that as a relational structure, X is a substructure of Y.

$$\forall s,t\ (s \in (S)_t \to s \in (\underset{\sim}{Y})_t) \ \&$$
$$(\forall s)(\forall t > 0)\ (s \in (\underset{\sim}{Y})_t \ \&\ \forall i < \ell h(s)((s)_i \in (S)_0)) \to s \in (S)_t$$

With parameters $\underset{\sim}{A}$ and $\underset{\sim}{c}$ which refer respectively to codes for the structure $A \in \overline{\mathcal{F}}_\omega$ and the partition $c:\binom{Y}{A} \longrightarrow r$, the clauses (i) through (iv) can be formalized without the use of second order quantifiers. Hence $\mathcal{A}$ is definable by a $\underset{\sim}{\Sigma}^1_1$ formula in the language $\underset{\sim}{L}^2_N$ and $\mathcal{A}$ is analytic. $\square$

<u>example 7.7</u> Let $Y \in \overline{\mathcal{K}}_\omega$ be the uniform object of $\deg(Y) = 2$ and $\mathcal{U} = \{S \subseteq Y |\ (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y \ \&\ \{\ell \in \omega |\ X \cap Y(\ell) \neq 0\} \in \mathcal{R} \ \&\ X \subseteq S)\}$, where $\mathcal{R}$ is a Ramsey ultrafilter. Then by Theorem 7.6

and Example 7.1 we have $\mathcal{U} \mapsto [\mathcal{U}]^2_{13}$.

Lemma 7.8 Let $\mathcal{R}$ be a Ramsey ultrafilter on ω. Given $Y \in \overline{\mathcal{K}}_\omega$ let $\mathcal{U} = \{S \subseteq Y: (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y$ & $\{\ell \in \omega \mid X \cap Y(\ell) \neq 0\} \in \mathcal{R}$ & $X \subseteq S)\}$. $\mathcal{U}$ is a countably complete co-ideal on Y.

proof: Given any $\mathcal{C}:\binom{Y}{Y\upharpoonright 1} \longrightarrow 2$, Theorem 7.6 gives $X \in \overline{\mathcal{K}}_\omega$ such that X reduces $\mathcal{C}$ and $\{\ell \in \omega: X \cap Y(\ell) \neq 0\} \in \mathcal{R}$ so $\mathcal{U}$ is a co-ideal on Y. Let $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$ be a decreasing sequence where $(\forall n \in \omega)(S_n \in \mathcal{U})$, and let $X_n \subseteq S_n$ satisfy $X_n \in \mathcal{U}$, $X_n \subset\subset Y$, and $\forall n \forall \ell$ $(X_{n+1} \cap Y(\ell) \neq 0 \rightarrow X_n \cap Y(\ell) \neq 0)$. We will find $X \in \mathcal{U}$ such that $X \subset\subset Y$ and $(\forall n \in \omega)(X - S_n \notin \mathcal{U})$. Define $f:\omega \longrightarrow \omega$ by induction beginning with $f(0) = m$ such that $X_0^{(0)} \subseteq Y^{(m)}$ and put $L_0 = \{\ell: X_0 \cap Y^{(m)} \cap Y(\ell) \neq 0\}$. Having $f(n)$ and L_n define $f(n+1) =$ the least m such that

(i) $m > f(n)$

(ii) $X_{n+1} \cap Y^{(m)} \neq 0$

(iii) $\{\ell: X_{n+1} \cap Y^{(m)} \cap Y(\ell) \neq 0\} \subsetneq L_n$.

Put $L_{n+1} = \{\ell: X_{n+1} \cap Y^{(m)} \cap Y(\ell) \neq 0\}$. Note $\forall n\ (L_n \in \mathcal{R}$ & $L_{n+1} \subsetneq L_n$ & $n \notin L_{n+1}$ & $n \leq f(n) < f(n+1))$. Since $\bigcap_{n\in\omega} L_n = 0$ and $L_n - L_{n+1} \neq 0$ we can define $g: L_0 \longrightarrow\!\!\!\!\rightarrow \omega$ for $t \in L_0$ by $g(t) = n$ such that $t \in L_n - L_{n+1}$. Define the partition $\mathcal{C}:[L_0]^2 \rightarrow 3$ for

$s,t \in L_0$ and $s < t$ by

$$c(\{s,t\}) = \begin{cases} 0 & \text{if } g(s) < g(t) \ \& \ s < f(g(t)) \\ 1 & \text{if } g(s) < g(t) \ \& \ s \geq f(g(t)) \\ 2 & \text{if } g(s) \geq g(t) \end{cases}$$

Let $L \in \mathcal{R}$ be homogeneous for c, and let $s \in L$ be the least member of L. If $c''[L]^2 = \{2\}$, then $(\forall t \in L - \{s\})(g(s) \geq g(t))$ so for some $n \leq g(s)$, $L_n - L_{n+1} \in \mathcal{R}$ contrary to $L_n, L_{n+1} \in \mathcal{R}$.

If $c''[L]^2 = \{1\}$, then $(\forall t \in L - \{s\})(s \geq f(g(t)))$ and since $f(g(t)) \geq (g(t))$ this means $s \geq g(t)$, which gives the same contradiction. Hence $c''[L]^2 = \{0\}$.

For any $s,t \in L$, we have $s \in L_{g(s)}$ and $t \in L_{g(t)}$ (by the definition of g) and if $s < t$ then $g(s) < g(t)$ so $L_{g(s)} \supsetneq L_{g(t)}$ gives $t \in L_{g(s)}$. That is $L - s \subseteq L_{g(s)}$ and for each $s \in L$ we can find $X^s \subset\subset X_{g(s)} \cap Y^{(f(g(s)))}$ (a sub-object in the sense of $\overline{\mathcal{K}}$) such that $\{\ell \in \omega | \ X^s \cap Y(\ell) \neq 0\} = L - s$. Note that these X^s are sub-objects of distinct columns, $Y^{(f(g(s)))}$ of Y, and the root of X^s is on level $Y(s)$. Let $X = \bigcup_{s \in L} X^s$ be structured by inheritance from Y. We claim

(i) $\{\ell \in \omega | \ X \cap Y(\ell) \neq 0\} = L \in \mathcal{R}$.

(ii) $(\forall x \in X)(\deg_X(x) = \deg_Y(x))$

(iii) $(\forall \ell \in L)(X \cap Y(\ell)$ contains exactly one root of X)

(iv) X is a sub-structure of Y

(v) $\forall n \ (X - S_n) \notin \mathcal{H}$.

(i) is clear.

(ii) follows from the sub-object condition $X^s \subset\subset X_{g(s)} \cap Y^{(f(g(s)))}$ on the columns of X.

A root of X is a root of some column of X so (iii) follows from $\{\ell \in \omega \mid X^s \cap Y(\ell) \neq 0\} = L - s$ and (i).

Since X is the union of columns X^s which use levels $L - s$, to show closure of X under Δ it suffices to show $(\forall x,y \in X)(x << y \ \& \ y \text{ is a root of } X \rightarrow x \Delta y = x)$. But $x << y$ means for some $s < t \in L$, $x \in Y(s)$ and $y \in Y(t)$. Since y is a root of X, $y \in X_{g(t)} \cap Y^{(f(g(t)))}$ and hence $y \in Y^{(f(g(t)))}$. But $s < t \rightarrow s < f(g(t))$ so $x \in Y(s) \ \& \ y \in Y^{(f(g(t)))} \rightarrow x \Delta y = x$. Since X is closed under Δ, X is a substructure of Y.

By the construction of X $X - S_n \subseteq \bigcup\{X^s : s \in L \ \& \ g(s) < n\}$ and since $(\forall s,t \in L)(s < t \rightarrow g(s) < g(t))$ this means $X - S_n$ is contained in a finite union of columns X^s so $X - S_n \notin \mathcal{H}$.

Condition (i) through (iv) show $X \in \mathcal{H}$ (using Lemma 6.9) and with condition (v) the countable completeness of has been demonstrated. □

<u>Theorem 7.9</u> (CH) There exists an ultrafilter $\mathcal{U}$ which satisfies $\mathcal{U} \longmapsto [\mathcal{U}]^2_{13}$.

<u>proof</u>: Lemma 7.8 shows that the co-ideal, $\mathcal{H}$, of Example 7.7 is countably complete and Theorem 5.10 gives $\mathcal{U} \subseteq \mathcal{H}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_{13}$. □

The basic technique which gave $\mathcal{U} \longmapsto [\mathcal{U}]^2_{13}$ above can be modified in many ways. Most trivially, the construction can be carried out in the context of a uniform object $Y \in \overline{\mathcal{K}}_\omega$ of deg(Y) = 3 (etc.) in which case an ultrafilter satisfying $\mathcal{U} \longmapsto [\mathcal{U}]^2_{32}$ (etc.) is produced. Additional properties of these ultrafilters can be discovered by looking more closely at the construction procedure.

We need some standard terminology for discussing ultrafilters on countable sets.

<u>Definition 7.10</u> Given an ultrafilter $\mathcal{U}$ on the countable set S and functions f and g with domain S ,

f is one-one (Mod $\mathcal{U}$) iff $(\exists X \in \mathcal{U})(f|X$ is one to one)

f is finite-one (Mod $\mathcal{U}$) iff $(\exists X \in \mathcal{U})(f|X$ is finite to one)

f is infinite-one (Mod $\mathcal{U}$) iff f is not finite-one (Mod $\mathcal{U}$)

$f \cong g$ (Mod $\mathcal{U}$) iff $(\exists X \in \mathcal{U})(\forall x,y \in X)(f(x) = f(y) \leftrightarrow g(x) = g(y))$

$f \preccurlyeq g$ (Mod $\mathcal{U}$) iff $(\exists X \in \mathcal{U})(\forall x,y \in X)(g(x) = g(y) \rightarrow f(x) = f(y))$

$f \prec g$ (Mod $\mathcal{U}$) iff $f \preccurlyeq g$ (Mod $\mathcal{U}$) and $f \not\cong g$ (Mod $\mathcal{U}$).

f is principal (Mod $\mathcal{U}$) iff $(\exists X \in \mathcal{U})(\forall x,y \in X)(f(x) = f(y))$

f is trivial (Mod $\mathcal{U}$) iff f is one-one (Mod $\mathcal{U}$) or f is principal (Mod $\mathcal{U}$)

$\mathcal{U}$ is a <u>p-point</u> iff every non-trivial function f on S is finite-one (Mod $\mathcal{U}$)

$\mathcal{U}$ is a <u>q-point</u> iff every finite-one function is trivial (Mod $\mathcal{U}$)

Given a map $f: S \longrightarrow S'$, the image of f,

$f(\mathcal{U}) = \{P \subseteq S' : f^{-1}(P) \in \mathcal{U}\}$, is an ultrafilter on S'.

Given an ultrafilter $\mathcal{V}$ on S', theRudin-Kiesler relation $\mathcal{V} \leq \mathcal{U}$ holds just in case there exists $f: S \to S'$ such that $f(\mathcal{U}) = \mathcal{V}$.

The ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are isomorphic iff there exists a bijection $f: S \hookrightarrow\!\!\!\!\rightarrow S'$ such that $f(\mathcal{U}) = \mathcal{V}$.

The notation $\mathcal{V} \prec \mathcal{U}$ indicates $\mathcal{V} \leq \mathcal{U}$ and $\mathcal{V} \not\cong \mathcal{U}$.

The Rudin-Kiesler relation, $\leq$, on the class of ultrafilters on ω (or on countable sets) is reflexive and transitive.

The minimal (non-principal) ultrafilters with respect to $\leq$ are the Ramsey ultrafilters.

$f \cong g \ (\mathrm{Mod}\,\mathcal{U}) \rightarrow f(\mathcal{U}) \cong g(\mathcal{U})$ but not conversely.

$f \prec g \ (\mathrm{Mod}\,\mathcal{U}) \rightarrow f(\mathcal{U}) \prec g(\mathcal{U})$

<u>Definition 7.11</u> The <u>level projection map</u>, $\pi_\ell : T_\omega \twoheadrightarrow \omega$, is defined for $t \in T_\omega$ by $\pi_\ell(t) = n$ such that $t \in T_\omega(n)$ (i.e., $\pi_\ell(t) = |t| - 1$). The <u>column projection map</u>, $\pi_c : T_\omega \twoheadrightarrow \omega$, is defined for $t \in T_\omega$ by $\pi_c(t) = m$ such that $t \in T_\omega^{(m)}$.

<u>Lemma 7.12</u> Let $Y \in \mathcal{K}_\omega$ be a fixed canonical object and let $\mathcal{Y} = \{S \subseteq Y \mid (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y \ \& \ X \subseteq S)\}$ Any ultrafilter $\mathcal{U}$ on Y such that $\mathcal{U} \subseteq \mathcal{Y}$ will be a non-p-point and a non-q-point.

<u>proof</u>: Since Y is canonical we can consider π_ℓ and π_c restricted to Y and its subsets. π_ℓ is finite-1 $(\mathrm{Mod}\,\mathcal{U})$

but not (1-1 Mod $\mathcal{U}$). π_c is infinite-1 (Mod $\mathcal{U}$) but not principal (Mod $\mathcal{U}$).

<u>Lemma 7.13</u> Let $Y \in \mathcal{K}_\omega$ be a uniform object with $\deg(Y) = d$ and let $\mathcal{Y} = \{S \subseteq Y \mid (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y \ \& \ X \subseteq S)\}$. Suppose an ultrafilter $\mathcal{U}$ on Y satisfies $\mathcal{U} \subseteq \mathcal{Y}$ and $(\forall A \in \overline{\mathcal{F}}_\omega)(A \subset\subset Y \ \& \ A$ is the closure of a pair in $Y \rightarrow$ for any finite partition $\kappa\!:\binom{Y}{A} \rightarrow r$, $\exists X \in \overline{\mathcal{K}}_\omega \cap \mathcal{U}$ such that X is homogeneous for κ.)

If f with domain S is non-trivial (Mod $\mathcal{U}$) then either $f \cong \pi_c$ (Mod $\mathcal{U}$) or $\pi_\ell \preccurlyeq f$ (Mod $\mathcal{U}$). $\pi_\ell(\mathcal{U})$ and $\pi_c(\mathcal{U})$ are Ramsey ultrafilters.

If $d = 1$ and f is non-trivial (Mod $\mathcal{U}$) then either $f \cong \pi_c$ (Mod $\mathcal{U}$) or $f = \pi_\ell$ (Mod $\mathcal{U}$).

If $d = 2$, f and g are non-trivial (mod $\mathcal{U}$) and $\pi_\ell \prec f$ (mod $\mathcal{U}$) and $\pi_\ell \prec g$ (Mod $\mathcal{U}$) then $f \cong g$ (Mod $\mathcal{U}$).

<u>proof</u>: Define $\kappa\!:[Y]^2 \longrightarrow 2$ for a pair $\{x,y\}$ by

$$\kappa(\{x,y\}) = \begin{cases} 0 & \text{if } f(x) = f(y) \\ 1 & \text{otherwise} \end{cases}$$

Since Y is uniform, there are only finitely distinct $A \in \mathcal{F}_\omega$ such that $A \subset\subset Y$ and A is the closure of some pair $\{x,y\} \in [Y]^2$. For each such $A \in \mathcal{F}_\omega$ a partition $\bar{\kappa}\!:\binom{Y}{A} \longrightarrow 2$ is naturally induced from $\kappa\!:[Y]^2 \longrightarrow 2$, and finitely many applications of our partition assumption gives $X \in \overline{\mathcal{K}}_\omega \cap \mathcal{U}$ such that for every such A, $\bar{\kappa}$ is constant on $\binom{X}{A}$.

<u>case 1</u>

Suppose there exist $x,y \in X$ with $x \ll y$ and $f(x) = f(y)$. Put $A = \Delta$-closure of $\{x,y\}$, so $\bar{c}''\binom{X}{A} = \{0\}$. Put $Z = \{z \in X : y \preccurlyeq z\}$ and note $(\forall z \in Z)((A - \{y\}) \cup \{z\} \in \binom{X}{A})$. But Z is a sub-object (in the sense of $\bar{\mathcal{K}}$) of some column of X. Since Y is uniform, so are X and Z with $\deg(Y) = \deg(X) = \deg(Z) = d$. Hence every possible type, B, of intra-column pair which ocurrs in Y (= the isomorphism type of the Δ-closure in the sense of $\bar{\mathcal{F}}_\omega$ or $\bar{\mathcal{F}}$) is represented by the closure of some pair $\{z,z'\} \in [Z]^2$. Since $f(z) = f(z')$, $\bar{c}''\binom{X}{B} = \{0\}$ and we conclude $(\forall x,y \in X)(\pi_c(x) = \pi_c(y) \rightarrow f(x) = f(y))$ so $f \preccurlyeq \pi_c$ (Mod $\mathcal{U}$).

If $f \prec \pi_c$ (Mod $\mathcal{U}$) then for some $x,y \in X$ we have $f(x) = f(y)$ and $\pi_c(x) = \pi_c(y)$. Using our case 1 assumption we can also assume this x,y satisfy $x \ll y$. But with $A = \Delta$-closure $\{x,y\}$ we note that every pair of columns $X^{(\ell)}$, $X^{(m)}$, $\ell \neq m$ has $x' \in X^{(\ell)}$ and $y \in X^{(m)}$ such that Δ-closure $\{x',y'\} \in \binom{X}{A}$, so $f(x') = f(y')$ and f is principal (Mod $\mathcal{U}$). Since we assumed f was non-trivial, $\pi_c \cong f$ (Mod $\mathcal{U}$).

<u>case 2</u> $(\forall x,y \in X)(x \ll y \rightarrow f(x) \neq f(y))$. That is $(\forall x,y \in X)(f(x) = f(y) \rightarrow \pi_\ell(x) = \pi_\ell(y))$ or $\pi_\ell \preccurlyeq f$ Mod $(\mathcal{U})$.

Suppose the f above is $f = g \circ \pi_c$ where $g : \omega \longrightarrow \omega$. Certainly for some $x,y \in X$ $(x \ll y$ & $g \circ \pi_c(x) = g \circ \pi_c(y))$ so the case 1 shows that $g \circ \pi_c$ is either trivial (Mod $\mathcal{U}$) or $g \circ \pi_c = \pi_c$ (Mod $\mathcal{U}$), so g is trivial (Mod $\pi_c(\mathcal{U})$) and $\pi_c(\mathcal{U})$ is Ramsey.

Suppose f above is $f = g \circ \pi_\ell$ where $g: \omega \longrightarrow \omega$. Then (assuming $g \circ \pi$ is non-trivial Mod $\mathcal{U}$) either $\pi_c = g \circ \pi_\ell$ (Mod $\mathcal{U}$) or $\pi_\ell \preccurlyeq g \circ \pi_\ell$ (Mod $\mathcal{U}$) But $\pi_c = g \circ \pi_\ell$ (Mod $\mathcal{U}$) is impossible and $\pi_\ell \preccurlyeq g \circ \pi_\ell$ (Mod $\mathcal{U}$) $\rightarrow$ g is one-one (Mod $\pi_\ell(\mathcal{U})$) so $\pi_\ell(\mathcal{U})$ is Ramsey.

Suppose $d = 1$ and $\pi_\ell \preccurlyeq f$ (Mod $\mathcal{U}$) where f is non-trivial (mod $\mathcal{U}$) and X is as above. Any pair $x,y \in X$ such that $\pi_\ell(x) = \pi_\ell(y)$ has the type $B_1 = \{\langle 0,0\rangle, \langle 1,0\rangle\} \in \mathcal{F}_\omega$ and since $d = 1$, and by our assumption on X, either $\bar{c}''\binom{X}{B_1} = \{0\}$ or $\bar{c}''\binom{X}{B_1} = \{1\}$. But $\bar{c}''\binom{X}{B_1} = \{1\}$ $\rightarrow$ f is one-one on X and hence trivial (Mod $\mathcal{U}$). Thus $\bar{c}''\binom{X}{B_1} = \{0\}$ and $\pi_\ell = f$ (Mod $\mathcal{U}$).

Suppose $d = 2$ and $\pi_\ell \prec f$ (Mod $\mathcal{U}$). Any $x \neq y \in X$ such that $\pi_\ell(x) = \pi_\ell(y)$ has possible type B_1 or $B_2 = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0\rangle\} = \wedge \in \mathcal{F}_\omega$. If $\bar{c}''\binom{X}{B_1} = \{0\}$ then $\bar{c}''\binom{X}{B_2} = 0$ since the $d = 2$ assumption implies the existence of distinct $x,y,z \in X$ such that $\{x,z\},\{y,z\} \in \binom{X}{B_1}$ and A-closure $\{x,y\} \in \binom{X}{B_2}$. Since $\pi_\ell \prec f$ (Mod $\mathcal{U}$) entails $\pi_\ell \not\equiv f$ (Mod $\mathcal{U}$) we conclude $\bar{c}''\binom{X}{B_1} \neq \{0\}$ and any non-trivial f such that $\pi_\ell \prec f$ (Mod $\mathcal{U}$) is characterized by $(\forall x,y \in X)\ (f(x) = f(y) \leftrightarrow \{x,y\} \in \binom{X}{B_2})$. $\square$

<u>Remark 7.14</u> Obviously the technique used to analyze the cases above with d = 1 or d = 2 is applicable also to the case d > 2 and similar results are obtainable.

<u>Theorem 7.15</u> (CH). Let $\mathcal{R}$ be a Ramsey ultrafilter on ω, and let $Y \in \mathcal{K}_\omega$ with
$\mathcal{G} = \{S \subseteq Y \mid (\exists X \in \overline{\mathcal{K}}_\omega)(X \subset\subset Y \ \& \ X \subseteq S \ \& \ \{\ell \in \omega \mid X \cap Y(\ell) \neq 0\} \in \mathcal{R}\}$.
There exists an ultrafilter $\mathcal{U}$ on Y such that $\mathcal{U} \subseteq \mathcal{G}$ and for any $A \in \overline{\mathcal{F}}_\omega$ and finite partition $\mathcal{C}: \binom{Y}{A} \longrightarrow r$ $\exists X \in \overline{\mathcal{K}}_\omega$ such that X is homogeneous for $\mathcal{C}$.

<u>Remark 7.16</u> Any such $\mathcal{U}$ is a non-p-point, non-q-point. If Y is also uniform then for any non-principal $\mathcal{V}$ $(\mathcal{V} \leq \mathcal{U} \rightarrow \mathcal{V} \cong \pi_c(\mathcal{U})$ or $\pi_\ell(\mathcal{U}) \leq \mathcal{V})$ and there is a function $\phi: \omega \longrightarrow \omega$ (depending on deg(Y)) such that $\mathcal{U} \mapsto [\mathcal{U}]^n_{\phi(n)}$. If $d = 1$, $\mathcal{U} \mapsto [\mathcal{U}]^2_5$. If $d = 2$, $\mathcal{U} \mapsto [\mathcal{U}]^2_{13}$.

<u>proof of 7.15</u>: There are only countably many $A \in \mathcal{F}_\omega$ such that $\binom{Y}{A} \neq 0$, so using CH let $\langle c_\gamma : \gamma \in \omega_1 \rangle$ be an enumeration of all finite partitions $\mathcal{C}: \binom{Y}{A} \longrightarrow r$ for all such $A \in \mathcal{F}_\omega$. Let A_γ denote the type of the object being partitioned by $\mathcal{C}_\gamma$. Let $\langle S_\gamma : \gamma \in \omega_1 \rangle$ be an enumeration of the subsets of Y. Just as in the proof of Theorem 5.10, a basis, $\langle B_\gamma : \gamma \in \omega_1 \rangle$, is constructed for $\mathcal{U}$ by induction.

Let $B_0 = Y$ and having B_γ we use Theorem 7.6 to find $B_{\gamma+1}$ such that

(i) $B_{\gamma+1} \in \mathcal{G}$

(ii) $B_{\gamma+1} \subseteq B_\gamma$

(iii) $\mathcal{C}_\gamma$ is constant on $\binom{Y}{A_\gamma}$

(iv) $B_{\gamma+1} \subseteq S_\gamma$ or $B_{\gamma+1} \subseteq Y - S_\gamma$.

For a limit ordinal $\lambda \in \omega_1$, having B_γ for $\gamma < \lambda$ use Lemma 7.8 to find $B_\lambda \in \mathcal{U}$ such that $(\forall\gamma \in \lambda)(B_\lambda - B_\gamma \notin \mathcal{U})$.

$\langle B_\gamma : \gamma \in \omega_1 \rangle$ is clearly the basis for an ultrafilter on Y which satisfies the requirements of the theorem. □

The combinatorial theorem used to produce the ultrafilters of Theorems 7.9 and 7.15 was essentially Theorem 6.15 although for countable completeness a strengthened version (Theorem 7.6) was actually used.

Recall that the motivation for stating Theorem 6.15 stemmed from consideration of a co-ideal $\mathcal{U} \otimes \mathcal{U}_0$ where $\mathcal{U} = [\omega]^\omega$ and $\mathcal{U}_0$ was a co-ideal on some $X \in \mathcal{K}$. $\mathcal{U} \otimes \mathcal{U}_0$ is automaticahly countably complete, so our attempt at formalizing the partition properties of $\mathcal{U} \otimes \mathcal{U}_0$ in terms of $\mathcal{C}_\omega$ and Theorem 6.15 has not entirely succeeded. The category $\mathcal{C}_\omega$ must be modified to deal with the combinatorial properties of $\mathcal{U} \otimes \mathcal{U}_0$. In Chapter 3 the category $\mathcal{C}$ and Theorem 2.10 were modified to produce categories $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ with their associated partition theorems. The situation now is analogous with $\mathcal{C}_\omega$ and Theorem 6.15 playing the role of $\mathcal{C}$ and Theorem 2.10.

Actually we are not so interested in modifying the category $\mathcal{C}_\omega$ as we are in the ultrafilters constructable from such modifications. Hence, the formalization of new categories will be omitted and we go directly to the construction of ultrafilters. Our constructions will give very specific examples from which similar results can be inferred using the same techniques.

<u>Lemma 7.17</u> Let $\mathcal{U}$ be the co-ideal $[\omega]^{\omega}$ and let $Y \in \mathcal{K}$ be the binary tree, $\deg(Y) = 2$. Let $\mathcal{U}_0 = \{S \subseteq Y: (\exists X \in \overline{\mathcal{K}})(X \subset\subset Y \ \& \ X \subseteq S)\}$. Then $\mathcal{U} \otimes \mathcal{U}_0 \longmapsto [\mathcal{U} \otimes \mathcal{U}_0]^2_{10}$.

<u>proof</u>: First we must show that for every $S \in \mathcal{U} \otimes \mathcal{U}_0$ and every partition $\kappa: [S]^2 \longrightarrow 11$ there exists $Z \in \mathcal{U} \otimes \mathcal{U}_0$ such that $|\kappa''[Z]^2| \leq 10$. But for any such S there is an injection $\psi: \omega \times Y \hookrightarrow S$ such that for any $P \in \mathcal{U} \otimes \mathcal{U}_0$, $\psi''P \in \mathcal{U} \otimes \mathcal{U}_0$. Using ψ^{-1}, the partition κ can be pulled back to a partition of $[\omega \times Y]^2$ and by following ψ^{-1} and ψ it clearly suffices to consider the case of a partition $\kappa: [\omega \times Y]^2 \rightarrow 11$.

Let $\overline{Y} \in \mathcal{K}_{\omega}$ be the unique object such that $\deg(\overline{Y}) = 2$, and let $\phi: \omega \times Y \hookrightarrow\!\!\!\rightarrow \overline{Y}$ be the unique bijection such that $(\forall n \in \omega)\Big((\phi \restriction_{\{n\} \times Y} \hookrightarrow\!\!\!\rightarrow \overline{Y}^{(n)}$ is a bijection$)$ &

$(\forall y,z \in Y)(y <^{Y} z \rightarrow \phi(n,y) <^{\overline{Y}} \phi(n,z))\Big)$. Following the correspondence $\phi: \omega \times Y \rightarrow \overline{Y}$ the partition $\overline{\kappa}: [\overline{Y}]^2 \longrightarrow 11$ is induced from c, and the co-ideal $\overline{\mathcal{U} \otimes \mathcal{U}_0} = \{\phi''S: S \in \mathcal{U} \otimes \mathcal{U}_0\}$ is induced on $\overline{Y}$. For any $A \in \overline{\mathcal{F}}_{\omega}$ such that A is the A-closure of some pair $\{x,y\} \in [\overline{Y}]^2$ we regard $\overline{\kappa}$ as a partition of $\binom{\overline{Y}}{A}$ in the usual way. Using Theorem 6.15 finitely many times we find $X \in \overline{\mathcal{K}}_{\omega}$ such that $\overline{X} \subset\subset \overline{Y}$ and for all $A \in \mathcal{F}_{\omega}$ as above, κ is constant on $\binom{\overline{X}}{A}$. As in Example 7.3 there are 13 possible types, $A \in \mathcal{F}_{\omega}$, of pair $\{x,y\} \in [\overline{X}]^2$, consisting of the 7 intra-column types and the 6 inter-column types. Note $\overline{X} \in \overline{\mathcal{U} \otimes \mathcal{U}_0}$.

Let $L = \{\ell \in \omega : \overline{X} \cap \overline{Y}(\ell) \neq 0\}$ and let $L = \bigcup_{n\in\omega} L_n$ be a disjoint decomposition of L into infinite sets L_n. For each n, choose some $\overset{n}{Z} \subset\subset \overline{X}^{(n)}$ (a sub-object in the sense of $\overline{\mathcal{K}}$) such that $(\overset{n}{Z} \cap \overline{Y}(\ell) \neq 0 \rightarrow \ell \in L_n)$ and $(\forall z \in \overset{n}{Z})(\forall x \in X - \overset{n}{Z})\ (x \prec z \rightarrow x \prec_0 z)$. Put $Z = \bigcup_{n\in\omega} \overset{n}{Z}$. Clearly $Z \in \overline{\mathcal{Y} \otimes \mathcal{Y}_0}$ and inter-column pairs of the three types

$\cdot\ \ \cdot\ = \{\langle 0,0\rangle, \langle 1,0\rangle\}$

$\cdot\ \ \diagdown\ = \Delta$-closure $\{\langle 0,0\rangle, \langle 1,0,1\rangle\}$

$\diagdown\ \ \cdot\ = \Delta$-closure $\{\langle 0,0,1\rangle, \langle 1,0\rangle\}$

do not occur in Z (see Example 7.1 for notation). In this context, the type of a pair from Z is of course being computed in $\overline{X} \in \overline{\mathcal{K}}_\omega$, as the isomorphism type of the Δ-closure in the sense of $\mathcal{F}_\omega$. Since κ is constant on the remaining $13 - 3 = 10$ types of pair which occur in Z, we have proved $\mathcal{Y} \otimes \mathcal{Y}_0 \longrightarrow [\mathcal{Y} \times \mathcal{Y}_0]^2_{11}$. Note that for any $Z \in \overline{\mathcal{Y} \otimes \mathcal{Y}_0}$, clearly all 7 intra-column types of pair are necessarily represented by a pair from Z and so is the inter-column type $\cdot\ ,\ = \{\langle 0\rangle, \langle 1,0\rangle\}$. It is also easily seen that at least one of the types from each of the sets

$\{\ \cdot\diagup\ ,\ \cdot\diagdown\ \}$

and

$\{\ \diagup\cdot\ ,\ \diagdown\cdot\ \}$

is represented in Z. This gives a classification of pairs $[\overline{Y}]^2$

into 7 intra-column and 3-intercolumn varieties which are essential to any $Z \in \overline{\mathcal{U} \otimes \mathcal{U}_0}$, and thus $\mathcal{U} \otimes \mathcal{U}_0 \longmapsto [\mathcal{U} \otimes \mathcal{U}_0]^2_{10}$. □

Remark 7.18 The method used to verify the partition property above is essentially the same as the method used to obtain the results of Chapter 3 but without defining a category here, the discussion is much less detailed and formal. It is clear how to build a category formalism to handle the partition properties of the above co-ideal. Such a formalism is perhaps necessary for an unambiguous statement of the partition properties of $\mathcal{U} \otimes \mathcal{U}_0$ with respect to n-tuples, $n > 2$. For the sake of brevity we will confine our attention to partitions of pairs where a detailed analysis is especially interesting because of the implications concerning the Rudin-Kiesler relation.

Theorem 7.19 (CH) Let $\bar{Y} \in \mathcal{K}_\omega$ be the uniform object with $\deg(\bar{Y}) = 2$. There is an ultrafilter $\mathcal{U}$ on $\bar{Y}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_{10}$. $\mathcal{U}$ is a non-p-point and a non-q-point. For any $f: \bar{Y} \longrightarrow \omega$, either f is trivial (Mod $\mathcal{U}$), $\pi_c \prec \pi_\ell \cong f$ (Mod $\mathcal{U}$) or $f \cong \pi_c \prec \pi_\ell$ (Mod $\mathcal{U}$). $\pi_c'(\mathcal{U})$ is a Ramsey ultrafilter.

proof: The co-ideal $\mathcal{U} \otimes \mathcal{U}_0$ or $\overline{\mathcal{U} \otimes \mathcal{U}_0}$ from the previous example and Theorem 5.10 immediately give an ultrafilter $\mathcal{U}$ on $\bar{Y}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_{10}$. The maps π_c and π_ℓ show that $\mathcal{U} \subseteq \overline{\mathcal{U} \otimes \mathcal{U}_0}$ is not p-point and not q-point respectively. Given $f\colon \overline{Y} \longrightarrow \omega$, the partition $c\colon [\overline{Y}]^2 \longrightarrow 2$ defined for $\{x,y\} \in |\overline{Y}|^2$ by

$$c(\{x,y\}) = \begin{cases} 0 & \text{if } f(x) = f(y) \\ 1 & \text{otherwise} \end{cases}$$

can be analyzed just as in Lemma 7.13 to prove $\pi_\ell \preccurlyeq f \pmod{\mathcal{U}}$ or $f \cong \pi_c \pmod{\mathcal{U}}$. But in this case, clearly $\pi_c \prec \pi_\ell \pmod{\mathcal{U}}$. Since $\deg(\overline{Y}) = 2$, for some $X \in \mathcal{U}$ every pair $\{x,y\} \in [X]^2$ such that $\pi_\ell(x) = \pi_\ell(y)$ must satisfy, (Δ-closure $\{x,y\}$ is isomorphic to $\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0\rangle\} = \Lambda \in \mathcal{F}_\omega$.) Hence for non-trivial $f \pmod{\mathcal{U}}$, $\pi_\ell \preccurlyeq f \pmod{\mathcal{U}} \to \pi_\ell \cong f \pmod{\mathcal{U}}$

so either $\pi_c \prec \pi_\ell \cong f \pmod{\mathcal{U}}$

or $f \cong \pi_c \prec \pi_\ell \pmod{\mathcal{U}}$.

$\pi_c(\mathcal{U})$ is Ramsey be the same argument used in Lemma 7.13. $\square$

The same techniques are now applied in the context of the partition results for the categories $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.

<u>Lemma 7.20</u> Let $Y_1 \in \mathcal{K}_1$ be the uniform object with $\deg(Y_1) = 2$. Let $Y_2 \in \mathcal{K}_2$ be the unique object with binary skeleton, $\dot{Y}_2 \in \overline{\mathcal{K}}_1$ such that $\deg(\dot{Y}_2) = 2$ (as in Example 3.17). Let $Y_3 \in \mathcal{K}_3$ be the unique object with binary skeleton $\dot{Y}_3 \in \overline{\mathcal{K}}_1$ such that $\deg(\dot{Y}_3) = 2$ (as in Example 3.26). Put

$$\mathcal{H}_1 = \{S \subseteq Y_1\colon (\exists X \in \overline{\mathcal{K}}_1)(X \subset\subset Y_1 \;\&\; X \subseteq S)\}$$
$$\mathcal{H}_2 = \{S \subseteq Y_2\colon (\exists X \in \overline{\mathcal{K}}_2)(X \subset\subset Y_2 \;\&\; X \subseteq S)\}$$
$$\mathcal{H}_3 = \{S \subseteq Y\colon (\exists X \in \overline{\mathcal{K}}_3)(X \subset\subset Y_3 \;\&\; X \subseteq S)\}.$$
$$\mathcal{H} = [\omega]^\omega$$

Then

$$\mathcal{U} \otimes \mathcal{U}_1 \longmapsto [\mathcal{U} \otimes \mathcal{U}_1]^2_7$$
$$\mathcal{U} \otimes \mathcal{U}_2 \longmapsto [\mathcal{U} \otimes \mathcal{U}_2]^2_6$$
$$\mathcal{U} \otimes \mathcal{U}_3 \longmapsto [\mathcal{U} \otimes \mathcal{U}_3]^2_5$$

proof-sketch: It suffices to only consider finite partitions $\pi: [\omega \times Y_i]^2 \longrightarrow r$ (as in Lemma 7.17). Let $\dot{Y}_1 \in \mathcal{K}$ be the unique object such that $\deg(\dot{Y}_1) = 2$, so (as in the proof of Theorem 3.6), a partition of $[Y_1]^2$ corresponds (trivially) to a partition of $[\dot{Y}_1]^2$, and by the same identification a partition of $[\omega \times Y_1]^2$ corresponds to a partition of $[\omega \times \dot{Y}_1]^2$. More to the point, partitions of $[\omega \times Y_i]^2$ correspond to partitions of $[\omega \times \dot{Y}_i]^2$ for $i = 1,2,3$ as in the proofs of Theorems 3.6, 3.16, 3.26.

With $\overline{Y} \in \mathcal{K}_\omega$ such that $\deg(\overline{Y}) = 2$, the natural correspondence between $\omega \times \dot{Y}_i$ and $\overline{Y}$ induces a partition $\overline{\pi}: [\overline{Y}]^2 \longrightarrow r$, which we regard as a partition of $\binom{\overline{Y}}{A}$ for each $A \in \overline{\overline{\mathcal{F}}}_\omega$ such that A is the Δ-closure of some pair $\{x,y\} \in [\overline{Y}]^2$. Let $\overline{X} \in \overline{\mathcal{K}}_\omega$ satisfy $\overline{X} \subset\subset \overline{Y}$ (in the sense of $\overline{\mathcal{K}}_\omega$) and for all $A \in \overline{\overline{\mathcal{F}}}_\omega$ as above, $\overline{\pi}$ is constant on $\binom{\overline{X}}{A}$. $\overline{X}$ has 6 inter-column types of pair which are reduced as in Lemma 7.17 to three inter-column types occurring in some $Z \subseteq \overline{X}$ where $(\forall m \in \omega)(Z \cap \overline{X}^{(m)} \in \overline{\mathcal{K}})$. The intra-column types of pair occurring in each $Z \cap \overline{X}^{(m)}$ are now reduced by some $W^m \subseteq Z \cap \overline{X}^{(m)}$ from 7 to 4, 3, or 2 types (respectively for

$i = 1,2,3$) by applying the techniques of Theorems 3.6, 3.16, 3.26. By following the skeleton attachment maps and the correspondences between $\omega \times Y_i$, $\omega \times \dot{Y}_i$, and $\overline{Y}$, the facts $|\tilde{c}''[W^m]^2| \leq 5 - i$ and $|\tilde{c}''[\bigcup_{m\in\omega} W^m]^2| \leq 8 - i$ translate back to $\mathcal{U} \otimes \mathcal{U}_i \longrightarrow [\mathcal{U} \otimes \mathcal{U}_i]^2_{9-i}$ and thus finally

$\mathcal{U} \otimes \mathcal{U}_i \longmapsto [\mathcal{U} \otimes \mathcal{U}_i]^2_{8-i}$. □

Remark 7.21 There is an obvious alternate proof to the one outlined above which appeals directly to the combinatorial results of Chapter 3 rather than to the proofs to handle intra-column pairs, while inter-column pairs can be handled by a new combinatorial argument. The advantage of translating all combinatorial steps back to the single Theorem 6.15 only becomes apparent when partitions of n-tuples, $n > 2$, are considered. The proof outlined above will deal with such partitions without further complication.

Theorem 7.22 (CH) Let $\overline{Y} \in \mathcal{K}_\omega$ be the uniform object with $\deg(\overline{Y}) = 2$, for $i = 1,2,3$ there exists an ultrafilter $\mathcal{U}_i$ on $\overline{Y}$ such that $\mathcal{U}_i \longmapsto [\mathcal{U}_i]^2_{8-i}$. $\mathcal{U}_i$ is a q-point and a non-p-point. π_ℓ is one-one (Mod $\mathcal{U}_i$) and $\pi_c(\mathcal{U}_i)$ is Ramsey. For any non-trivial f (Mod $\mathcal{U}_i$), $f = \pi_c$ (Mod $\mathcal{U}_i$).

proof-sketch: Let $Y \in \mathcal{K}$ be the uniform object with $\deg(y) = 2$. Using the notation of Lemma 7.20, we have $Y_i \subseteq Y$ for $i = 1,2,3$. Let $\phi : \omega \times Y \longrightarrow \overline{Y}$ be the unique bijection which satisfies

$(\forall n \in \omega)\Big((\phi \upharpoonright \{n\} \times Y \longleftrightarrow Y^{(n)}$ is a bijection &

$(\forall y,z \in Y)(y <^{Y} z \leftrightarrow \phi(n,y) <^{\overline{Y}} \phi(n,z)\Big)$.

For each $i = 1,2,3$ the restriction of ϕ to $\omega \times Y_i$ induces a co-ideal $\overline{\mathcal{J} \otimes \mathcal{J}}_i = \{P \subseteq \overline{Y}: (\exists S \in \mathcal{J} \otimes \mathcal{J}_i)\ \phi''S \subseteq P\}$ on $\overline{Y}$ which corresponds to the co-ideal $\mathcal{J} \otimes \mathcal{J}_i$ on $\omega \times Y_i$. Using $\overline{\mathcal{J} \otimes \mathcal{J}}_i \longmapsto [\overline{\mathcal{J} \otimes \mathcal{J}}_i]^2_{8-i}$ and Theorem 5.10 we have ultrafilters $\mathcal{U}_i \subseteq \overline{\mathcal{J} \otimes \mathcal{J}}_i$ such that $\mathcal{U}_i \longmapsto [\mathcal{U}_i]^2_{8-i}$. Clearly π_ℓ is one-one (Mod $\mathcal{U}_i$). Given any $f: \overline{Y} \to 2$ which is non-trivial (Mod $\mathcal{U}_i$) let $\kappa: [\overline{Y}]^2 \to 2$ be defined for $\{x,y\} \in [\overline{Y}]^2$ by

$$\kappa(\{x,y\}) = \begin{cases} 0 & \text{if } f(x) = f(y) \\ 1 & \text{otherwise} \end{cases}$$

The usual analysis of κ with respect to every possible type of pair leads to the conclusion $f \cong \pi_c$ (Mod $\mathcal{U}_i$) (the analysis of an intra-column pair $\{x,y\}$ in $\mathcal{J} \otimes \mathcal{J}_3$ such that $\kappa(\{x,y\}) = 0$ must be handled slightly differently than in Lemma 7.13).

Since π_c is the only non-trivial map (Mod $\mathcal{U}_i$), and π_c is infinite-one (Mod $\mathcal{U}_i$), $\mathcal{U}_i$ is a non-p-point and a q-point. □

<u>Theorem 7.23</u> (CH) There exists an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_4$ which is a q-point and non-p-point. There is a map, π, such that $\pi(\mathcal{U})$ is Ramsey and every non-trivial f (Mod $\mathcal{U}$) satisfies $f \cong \pi$ (Mod $\mathcal{U}$).

proof: Let $Y \varepsilon \mathcal{K}_1$ have $\deg(Y) = 1$ and put $\mathcal{H}_4 = \{S \subseteq Y: (\exists X \varepsilon \bar{\mathcal{K}}_1)(X \subset\subset Y \ \& \ X \subseteq S)\}$. Analyze $\mathcal{H} \otimes \mathcal{H}_4$, and an ultrafilter $\mathcal{U} \subseteq \mathcal{H} \otimes \mathcal{H}_4$ as in Lemma 7.20 and Theorem 7.22. □

All of the ultrafilters constructed thus far have been non-p-points (except for the Ramsey ultrafilters) because our co-ideals have forced π_c to be infinite-one (Mod $\mathcal{U}$). Partition theorems for finite objects can be used to produce p-point ultrafilters.

The proto-type for such constructions is the well known construction of a p-point ultrafilter which is not Ramsey.

Lemma 7.24. Let $S = \bigcup_{n\varepsilon\omega} \{n\} \times (n + 1)$ and let $\mathcal{H} = \{Z \subseteq S: \forall m \, \exists n > m \ |(\{n\} \times (n + 1)) \cap Z| > m\}$. $\mathcal{H}$ is a co-ideal and satisfies $\mathcal{H} \longmapsto [\mathcal{H}]^2_3$.

proof: We will refer to $\{n\} \times (n + 1)$ as the n'th-column of S and $(\omega \times \{m\}) \cap S$ as the m'th level of S. Given a partition $\kappa: [S]^1 \longrightarrow 2$, each column of S can be reduced to the nodes of the majority color in each column and then an infinite set of reduced columns of the same color can be found thus showing that $\mathcal{H} \longrightarrow [\mathcal{H}]^1_2$ and $\mathcal{H}$ is a co-ideal.

Given a partition $\kappa: [S]^2 \longrightarrow 2$, S can be reduced by a simple induction to a set $Z \varepsilon \mathcal{H}$, $Z \subseteq S$ such that all intercolumn pairs from Z are the same color. The idea is simply to fix a point x, say in column n, and reduce columns n', $n' > n$ so that x paired with any of these subsequent points is the same color. By considering the points in each column inductively beginning with $\{0\} \times 1$, a set $Z \in \mathcal{U}$ and a partition $\bar{\kappa}: [Z]^1 \to 2$ are produced such that for any (n,m) and (n', m') in distinct columns of Z (i.e., $n \neq n'$, say $n < n'$) we have $\kappa(\{(n,m), (n',m')\}) = \bar{\kappa}(n)$. Let $\bar{Z} \in \mathcal{U}$ reduce $\bar{\kappa}$ and we have $\bar{\kappa}$ is constant on $[\bar{Z}]^1$ so κ is constant on all inter-column pairs from $\bar{Z}$. We note that the finiteness of the columns and the co-ideal property of $\mathcal{U}$ were the only facts used to prove that there is just one inter-column type of pair.

The columns of $\bar{Z}$ are now reduced using the finite Ramsey theorem for partitions of pairs to get $W \subseteq \bar{Z}$, $W \in \mathcal{U}$ such that κ is constant on the pairs from each column of W. Thus a color is associated with each column of W, and by taking an infinite set of columns of the same color finally we have $\bar{W} \subseteq W$, $W \in \mathcal{U}$ such that all inter-column pairs are the same color and all intra-column pairs are the same color. That is $\mathcal{U} \longrightarrow [\mathcal{U}]^2_3$, and by an obvious counter-example $\mathcal{U} \not\to [\mathcal{U}]^2_2$ so $\mathcal{U} \longmapsto [\mathcal{U}]^2_2$. $\square$

Corollary 7.25 Assuming CH, there exists a p-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_2$.

proof: Countable completeness of $\mathfrak{D}$ (from above) is shown by forming a set $Z \in \mathfrak{D}$ by a simple diagonal selection of columns from a descending countable sequence from $\mathfrak{D}$. Theorem 5.10 gives $\mathcal{U} \subseteq \mathfrak{D}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_2$ and the fact that $\mathcal{U}$ is p-point follows from this partition property. □

To generalize this construction we can replace the use of the finite Ramsey theorem by different partition results for finite objects.

Theorem 7.26 Given $A, B \in \overline{\mathfrak{F}}$ and $r \in \omega$ there exists $C \in \overline{\mathfrak{F}}$ such that for every partition $\mathcal{c}: \binom{C}{A} \longrightarrow r$ there exists $\overline{B} \in \overline{\mathfrak{F}}$ which satisfies $\overline{B} \subset\subset C$, $\overline{B}$ is isomorphic to B and $\mathcal{c}$ is constant on $\binom{\overline{B}}{A}$.

proof: Apply a compactness argument to Theorem 2.10. The technique is illustrated by the proof of Theorem 1.4. □

Let $Y \in \mathcal{K}$ be fixed and put $R = \bigcup_{n \in \omega} \{n\} \times Y \upharpoonright n+1$. Note that $S = \bigcup_{n \in \omega} \{n\} \times (n+1)$ can be regarded as the special case of R which results from a unary tree, Y. Put $\mathcal{B} = \{B \subseteq R: \forall m (\exists n > m)((\{n\} \times Y \upharpoonright n+1) \cap B \text{ is isomorphic to } Y \upharpoonright m)\}$.

The notion of isomorphism here results from viewing each column $\{n\} \times Y \upharpoonright n+1$ as an object of $\mathcal{F}$. Let $\mathcal{H} = \{Z \subseteq R : \exists B \in \mathcal{B} \ (B \subseteq Z)\}$.

Given any partition $\kappa : [R]^1 \longrightarrow 2$, Theorem 7.26 can be used on each column to find $Z \in \mathcal{B}$ such that κ is constant on each column. As in the proof of Lemma 7.24 we then find $\overline{Z} \subseteq Z$, $Z \in \mathcal{H}$ such that κ is constant on $[\overline{Z}]^1$. Thus $\mathcal{H}$ is a co-ideal.

Given any $\kappa : [R]^2 \longrightarrow 2$ there exists $\overline{Z} \in \mathcal{B}$ such that κ is constant on all inter-column pairs (as in Lemma 7.24). The number of intra-column pairs depends on the specific object $Y \in \mathcal{K}$ which gave rise to $\mathcal{H}$. For the uniform object, $\deg(Y) = 2$, we have 7 intra-column types of pair (as in example 2.2). Thus $\mathcal{H} \longmapsto [\mathcal{H}]^2_8$.

<u>Theorem 7.27</u> (CH) There exists an ultrafilter $\mathcal{U}$ which satisfies $\mathcal{U} \longmapsto [\mathcal{U}]^2_8$.

<u>proof</u>: The co-ideal $\mathcal{H}$ above is countably complete and yields our ultrafilter as usual. □

Actually much more can be said about the ultrafilter $\mathcal{U}$ of Theorem 7.27. In analogy to Theorem 7.19 the functions $f : R \longrightarrow \omega$ can be analyzed mod $\mathcal{U}$. Using our standard technique we find that $\mathcal{U}$ has a Ramsey ultrafilter as an image, $\mathcal{U}$ is a p-point etc.

This p-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_8$ is the counterpart (using a partition theorem for finite objects) of the ultrafilter of Theorem 7.19. As a counterpart to Theorem 7.22 we have:

Theorem 7.28 (CH) For $i = 1,2,3$ there exist p-point ultrafilters $\mathcal{U}_i$ such that $\mathcal{U}_i \longmapsto [\mathcal{U}_i]^2_{6-i}$.

proof: The finite partition theorem (7.26) for $\overline{\mathcal{F}}$ is true also for $\overline{\overline{\mathcal{F}}}_1$, $\overline{\overline{\mathcal{F}}}_2$, and $\overline{\overline{\mathcal{F}}}_3$ (and $\overline{\overline{\mathcal{F}}}_\omega$) by the same compactness argument so the construction proceeds just like Theorem 7.27. The number of intra-column types of pair is dependent on the context $\overline{\overline{\mathcal{F}}}_1$, $\overline{\overline{\mathcal{F}}}_2$ or $\overline{\overline{\mathcal{F}}}_3$ just as in Theorem 7.22, while finiteness of the columns gives just one inter-column type. The p-point property follows from the usual analysis of functions Mod $\mathcal{U}_i$. $\square$

Although we have been focusing attention on partitions of pairs it is clear how to build p-point ultrafilters with partition properties for n-tuples, $n > 2$. For example, if $Y \in \mathcal{K}_3$ with binary skeleton and the triples in $R = \bigcup_{n \in \omega} \{n\} \times Y \upharpoonright n$ are analyzed (by computing the type in the sense of $\mathcal{K}_3$) we find 5 inter-column types of triple and 16 intra-column types. With a co-ideal $\mathcal{H}$ on R defined as usual we have $\mathcal{H} \longmapsto [\mathcal{H}]^3_{21}$ and (with CH) a p-point ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^3_{21}$. In fact p-point ultrafilters with simultaneous partition properties for all n-tuples, $n \in \omega$, can be constructed as in Theorem 7.15.

Our examples of ultrafilter constructions using the partition properties of various 'tree-like' objects has demonstrated the existence of ultrafilters with a great variety of properties. The techniques of construction have not been presented in a completely uniform manner, however, and many variations on the definitions of the categories $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$ etc. easily come to mind. There are many questions. What is common to all of these categories and their partition theorems? What are the limits to the basic technique being illustrated by our examples? Can an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \longmapsto [\mathcal{U}]^2_n$ be constructed by these methods for every $n \in \omega$?

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