ABCDE Problem
Posed by an Anonymous Commenter
Here is William Gasarch's Solution

In the following problem A,B,C,D,E are digits. An expression like AB means the 2-digit number with an A in the tens place and a B in the units place.

1. Give all solutions to

   $ABCDE + BCDE + CDE + DE + E = 20320$

2. Show there are no other solutions.

All $\equiv$ are mod 10.

Theorem 0.1

1. *$E \in \{0, 2, 4, 6, 8\}$.*
2. *If $E = 0$ then*

$$(A, B, C, D, E) = (2, 0, 0, 8, 0)$$

   or

$$(A, B, C, D, E) = (1, 5, 0, 8, 0).$$

3. *$E \neq 2, 6$.*

Proof:
1) The E-sum says that $5E \equiv 0$. Hence E is even.

2) Assume $E = 0$. Then the D-carry is 0. Hence
$4D \equiv 2$
$4D - 2 \equiv 0$
$2(2D - 1) \equiv 0$.
So $2D - 1 \in \{5, 15\}$, so $D = 3, 8$.
Case 1: $D = 3$
$4D = 12$, so the C-carry is 1. Hence

$3C + 1 \equiv 3$
$3C \equiv 2$
$C \equiv 4$. Since $3C = 12$, the B-carry is 1. Hence
$2B + 1 \equiv 0$. This is impossible.

Case 2: $D = 8$

$4D = 32$, so the C-carry is 3. Hence
$3C + 3 \equiv 3$
$3C \equiv 0$
$C = 0$. So the B-carry is 0. Thus
$2B \equiv 0$
$B = 0$ or $B = 5$.
If $B = 0$ then the A-carry is 0, so $A = 2$.
If $B = 5$ then the A-carry is 1, so $A = 1$.
Hence

$$(A, B, C, D, E) = (2, 0, 0, 8, 0)$$

or

$$(A, B, C, D, E) = (1, 5, 0, 8, 0)$$

3) If $E \in \{2, 6\}$ then $E = 4x + 2$ with $x \in \{0, 1\}$. Hence
$5E = 20x + 10$. Hence the D-carry, which we denote d, is 1 or 3. Thus
$4D + d \equiv 2$. This is impossible since d is odd. ▮

The only cases of E to look at now are $E \in \{4, 8\}$.

Theorem 0.2

1. *If* $E = 4$ *then*

$$(A, B, C, D, E) = (2, 0, 1, 0, 4)$$

or

$$(A, B, C, D, E) = (1, 5, 1, 0, 4)$$

or

$$(A, B, C, D, E) = (1, 4, 7, 5, 4)$$

2. If $E = 8$ then

$$(A, B, C, D, E) = (1, 5, 0, 7, 8)$$

or

$$(A, B, C, D, E) = (2, 0, 0, 7, 8)$$

Proof:
1) $E = 4$: $5E = 20$ so the D-carry is 2.
$4D + 2 \equiv 2$
$4D \equiv 0$. So either $D = 0$ or $D = 5$.
Case $D = 0$.
$4D + 2 = 2$. C-Carry is 0. Hence
$3C = 3$. So $C = 1$. Carry is 0. Hence
$2B = 0$. So $B = 0$ or $B = 5$.
$B = 0$: No carry so $A = 2$. Hence

$$(A, B, C, D, E) = (2, 0, 1, 0, 4)$$

$B = 5$: Carry is 1 so $A = 1$. Hence

$$(A, B, C, D, E) = (1, 5, 1, 0, 4)$$

Case $D = 5$
$4D + 2 = 22$ so the C-carry is 2.
$3C + 2 \equiv 3$
$3C \equiv 1$
$C = 7$: $3C + 2 = 23$ so B-carry is 2.
$2B + 2 \equiv 0$
$2(B + 1) \equiv 0$
$B = 4$ OR $B = 9$.
If $B = 4$ then $2B + 2 = 10$ so A-carry is 1. Hence $A = 1$ so:

$$(A, B, C, D, E) = (1, 4, 7, 5, 4)$$

If $B = 9$ then $2B + 2 = 20$ so A-carry is 2. Hence $A = 0$ which is not allowed.

2) $E = 8$: $5E = 40$ so D-carry is 4

$4D + 4 \equiv 2$

$4D \equiv 8$

$4(D - 2) \equiv 0$. So $D = 2$ OR $D = 7$.

Case $D = 2$

$4D + 4 = 12$ so C-carry is 1.

$3C + 1 \equiv 3$

$3C \equiv 2$ so $C = 4$.

$C = 4$: $3C + 1 = 13$ so B-carry is 1.

$2B + 1 \equiv 0$.

$2B \equiv 9$. This has no solution.

Case $D = 7$

$4D + 4 = 32$ so C-carry is 3.

$3C + 3 \equiv 3$

$3C \equiv 0$ so $C = 0$. Carry is 0.

$2B \equiv 0$. So $B = 0$ or $B = 5$.

Case $B = 0$ The A-carry is 0 so $A = 2$. Hence:

$$(A, B, C, D, E) = (2, 0, 0, 7, 8)$$

Case $B = 5$ The A-carry is 1 so $A = 1$. Hence:

$$(A, B, C, D, E) = (1, 5, 0, 7, 8)$$

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From the above theorems we have the following:

Theorem 0.3 *The only solutions are*

1. $(A, B, C, D, E) = (2, 0, 0, 8, 0)$. *This is from Theorem 0.1.*
2. $(A, B, C, D, E) = (1, 5, 0, 8, 0)$. *This is from Theorem 0.1.*
3. $(A, B, C, D, E) = (2, 0, 1, 0, 4)$. *This is from Theorem 0.2.*
4. $(A, B, C, D, E) = (1, 5, 1, 0, 4)$. *This is from Theorem 0.2.*
5. $(A, B, C, D, E) = (1, 4, 7, 5, 4)$. *This if from Theorem 0.2.*
6. $(A, B, C, D, E) = (1, 5, 0, 7, 8)$. *This is from Theorem 0.2.*
7. $(A, B, C, D, E) = (2, 0, 0, 7, 8)$. *This is from Theorem 0.2.*