The smallest equations I cannot solve

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Motivated by Mathoverflow question
https://mathoverflow.net/questions/316708/what-is-the-smallest-unsolved-diophantine-equation

I have recently became interested in solving Polynomial Diophantine equations, that is, equations in the form

$P(x_1, \dots, x_n) = 0$ for some polynomial P. Because there are many such equations, I have decided to ask computer to help me. Our conversation is presented below.

Me: Can you help me solve Polynomial Diophantine equations? I will type in an equation and you will output all solutions.

Computer: Ok, but if there are infinitely many solutions I may run forever...

Me: Hm... Ok then. Let us solve an easier problem. I will type in an equation and you will output whether it has any integer solutions or not.

Computer: I can manage output a simple bit in finite time, so no problem. Just give me an algorithm how to decide this Yes/No question and let us start.

Me: Unfortunately, Matiyasevich proved in 1970 that there is no such algorithm.

Computer: Sorry, no money no honey. I mean - no algorithm no work.

Me: That's unfortunate... Ok, let us solve together at least small Diophantine equations.

Computer: What do you mean by "small"?

Me: Not sure yet. Some people define heigh of the equation as the largest absolute value of its coefficients. Can you list Polynomial Diophantine equations of height 0?

Computer: Sure. $0 = 0$.

Me: Nice! Let us move to height 1.

Computer: Ok. $1 = 0$, $-1 = 0$, $x_1 = 0$, $-x_1 = 0$, $x_1^2 = 0$, $-x_1^2 = 0$, $x_1^3 = 0, \dots$

Me: Wait! Stop! It looks like you will run forever this way. We need a better way to define the size of an equation, so that we have only a finite number of equations of any given size. Let us define size $H(P)$ of the equation $P(x_1, \dots, x_n) = 0$ in the following way: substitute 2 instead of all x_i, absolute values instead of all coefficients of P, and evaluate. For example, the equation $x^3 - y^2 - 3 = 0$ has size $H = 2^3 + 2^2 + 3 = 15$. Can you now list equations of size $H = 0, 1, 2, \dots$

Computer: Sure. One equation $0 = 0$ for $H = 0$, two equations $\pm 1 = 0$ for $H = 1$. Fot $H = 2$, we have $2n + 2$ equations: $\pm 2 = 0$ and $\pm x_i = 0$, $i = 1, \dots, n$. Then for $H = 3$ we have $\pm x_i \pm 1 = 0$, $i = 1, \dots, n$, and...

Me: Stop! You output the same equations many times! Let us call equations equivalent if one can be transformed into another after multiplication by -1, replacing some x_i by $-x_i$, and permutation of variables. Then please output only one equation from each equivalence class! Also, use letters $x, y, z, \dots$ instead of $x_1, x_2, x_3 \dots$.

Computer: Ok. Then we have equation $0 = 0$ for $H = 0$, $1 = 0$ for $H = 1$, $2 = 0$ and $x = 0$ for $H = 2$, $3 = 0$ and $x + 1 = 0$ for $H = 3$...

Me: That's better, but out of 6 equations you outputted me so far, 4 have no variables at all! Can you solve these equations yourself? I can give you an algorithm if you want - the answer is Yes for $0 = 0$ and No for $1 = 0$, $2 = 0$, and all further equations with no variables. Solve it yourself and do not bother me with such "equations" any more!

Computer: Ok. If we exclude equations with no variables, we get equation $x = 0$ for $H = 2$, $x + 1 = 0$ for $H = 3$, $x + 2 = 0$, $2x = 0$, $x^2 = 0$, $x + y = 0$ and $xy = 0$ with $H = 4$...

Me: Stop! These equations have variables, but still not interesting. All of them has some small integer solutions! For example, $x + 2 = 0$ has solution $x = -2$, while $x + y = 0$ has solution, for example,

$x = y = 0$. So the answer to our question is "Yes" for all such equations. Can you check all candidate solutions up to absolute value, say, 100, and do not output uninteresting equations that have small solutions?

Computer: Sure. Then I have no equations to output with $H \leq 4$. The smallest equations I now output are $x^2+1=0$ and $2x+1=0$ with $H=5$. Then...

Me: Stop! Your equations are still not interesting! The first one has no real solutions, while the second one has no solutions modulo 2. Of course, such equations cannot have integer solutions. This is called the integral Hasse principle. Moreover, there are well-known algorithms for recognizing equations with no real solutions or no solutions modulo some integer. Can you please check this and not output such equations?

Computer: Ok. Then have a look at the equation $x(y^2+2)=1$ with $H=13$. This one has real solutions and solutions modulo every integer, but I cannot find any integer solutions up to 100...

Me: This is because y^2+2 is never a divisor of 1! Will you ever output me something interesting?

Computer: What about equation $y^2 = x^3 - 3$ with $H=15$?

Me: Finally the first equation that is not completely trivial! However, this one is well-known! Equations in the form $y^2 = x^3 + k$ are called Pell equations, and they have been solved for all small k. In particular, it is known that there are no solutions for $k=-3$. In fact, there is an algorithm which, given k, determines all integer solutions to $y^2 = x^3 + k$. Moreover, the same is true for general cubic equation in 2 variables. So, please exclude all such equations and not output them.

Computer: Ok. Now the smallest equation for output is

$$y(x^2-y) = z^2+1$$

with $H=17$.

Me: Finally the first equation which is neither completely trivial nor well-known, at least to me. However, it is also not very difficult. It is known that all odd prime factors of z^2+1 must be of the form $p=4k+1$. Hence, if z^2+1 is odd, then it has only such prime factors. This implies that both y and x^2-y are 1 modulo 4, hence x^2 is 2 modulo 4, a contradiction. The case when z^2+1 is even is only a bit more difficult.

Computer: I am afraid it will be difficult for me to automatically finish the proof and exclude all further equations solvable by this method...

Me: No problem! You finally started to output some interesting equations! Please continue excluding the trivial and well-known equations, and I will manually solve the remaining ones as long as I can.

With computer, we then continue to work like this. Some equations, like

$$x^2+y^2-z^2 = xyz-2$$

with $H=22$ or

$$y(x^3-y) = z^2+2$$

with $H=26$ I could not solve myself and asked for help on Mathoverflow website. Finally, we have met an equation that even smart Mathoverflow users cannot solve for several weeks. The equation is

$$x(x^2+y^2+1) = z^3-z+1.$$

with $H=29$.

With computer, we have also studied restricted families of equations parametrized by degree, the number of variables, or the number of monomials. If parametrized by degree, there is a general method to solve all quadratic equations, while the smallest open cubic equation is listed above. For two-variable equations, we solved all with $H \leq 31$, but have several open ones with $H=32$, for example,

$$y^3+xy+x^4+4=0,$$

If parametrized by the number of monomials, equations with one or two monomials are easy, so the first interesting case is 3 monomials. In this family, we were able to solve all equations up to $H \leq 50$ with two exceptions: equation

$$x^3y^2 = z^3+6$$

with $H = 46$, and equation

$$x^3y^2 = z^4 + 2$$

with $H = 50$.

The readers are invited to try to solve these equations, and look at my recent arxiv preprint

https://arxiv.org/abs/2108.08705

and mathoverflow question

https://mathoverflow.net/questions/400714/can-you-solve-the-listed-smallest-open-diophantine-equations

for more details.