Ford vs Erdos-Szemeredi

1 Ford's Theorem

Theorem If $A = \{1, \ldots, n\}$ then $A \cdot A = \Theta\left(\frac{n^2}{(\log n)^c \log\log n)^{3/2}}\right)$ where $c \sim 0.086$.

Since $A + A = \Theta(n)$ we restate as follows:

Theorem If $A = \{1, \ldots, n\}$ then $\max\{A + A, A \cdot A\} \le c\frac{n^2}{(\log n)^c \log\log n)^{3/2}}\Big)$.

2 Erdos-Szemeredi Theorem

Theorem There is a constant d such that, for all n, there is a set $A \subset Z$, $|A| = n$ such that $\max\{A + A, A \cdot A\} \le n^2 \exp\left(\frac{-\ln n}{d \ln\ln n}\right)$

Note that

$$n^2 \exp\left(\frac{-\ln n}{d \ln\ln n}\right) = \frac{n^2}{\exp\left(\frac{\ln n}{d \ln\ln n}\right)} = \frac{n^2}{n^{1/d \ln\ln n}} = n^{2-\frac{1}{d \ln\ln n}}.$$

3 E-S Result is Better Than Fords

We show E-S is a better result. We ignore constants. We need:

$$= n^{2-\frac{1}{\log\log n}} \le \frac{n^2}{(\log n) \log\log n)^{3/2}}$$

$$(\log n)(\log\log n)^{3/2} n^{2-\frac{1}{\log\log n}} \le n^2.$$

$$(\log n)(\log\log n)^{3/2} \le n^{1/\log\log n}.$$

$$\log\log n + \frac{3}{2}\log\log\log n \le \frac{\log n}{\log\log n}$$

$$(\log\log n)^2 + \frac{3}{2}(\log\log\log n)(\log\log n) \le \log n$$

True for large n.