The Prove of a Well Known Phenomenon

Let $f\colon \mathbb{N} \rightarrow \mathbb{N}$ maps a natural to the number of letters in the word for that natural.

Here are the first 10 values:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	4	3	3	5	4	4	3	5	5	4	3

Facts I need

1. For all $a \le 3$, $f(a) > a$. This is easily proven by looking at $a = 0, 1, 2, 3$.
2. For all $a \ge 5$, $f(a) < a$. This would need to be stated more carefully; however, it would be a proof by induction which might need a large base case.

Theorem 0.1 *For all $a \in \mathbb{N}$, $a, f(a), \ldots$ is eventually all 4's.*

Proof:

Case 1: $a = 0, 1, 2$ or 3.

$0 \rightarrow 4$ DONE.

$1 \rightarrow 3 \rightarrow 5 \rightarrow 4$ DONE.

$2 \rightarrow 3 \rightarrow 5 \rightarrow 4$ DONE.

$3 \rightarrow 5 \rightarrow 4$ DONE.

Case 2: $a = 4$. $4 \rightarrow 4$. DONE.

Case 3: $a \ge 5$. By the fact above $a, f(a), \ldots$ will start out strictly decreasing. If it hits 4 (e.g., if it hits 5 or 9 and then hits 4) we are done. If not then it must go BELOW 4 and then you are at 0,1,2 or 3 and, by the reasoning in Part 1, will hit 4. ▮