The Roots Hierarchy

Exposition by William Gasarch and Erik Metz

1 Introduction

The main proof in this note is from *Problems from the Book* by Dospinescu and Andreescu.

We want to classify real numbers in terms of their complexity.

Def 1.1 Let $d \in \mathsf{N}$.

1. $\mathsf{Z}_d[x]$ is the set of polynomials of degree d over Z (the integers).

2. roots_d is the set of roots of polynomials in $\mathsf{Z}_d[x]$. Note that $\text{roots}_1 = \mathsf{Q}$.

Clearly

$$\text{roots}_1 \subseteq \text{roots}_2 \subseteq \text{roots}_3 \subseteq \cdots$$

We want to show that

$$\text{roots}_1 \subset \text{roots}_2 \subset \text{roots}_3 \subset \cdots$$

2 $7^{1/3} \notin \text{roots}_2$

In *Problems from the Book* by Dospinescu and Andreescu they show that, for all d, $2^{1/d}$ does not satisfy any $d-1$-degree polynomial over the rationals. Hence they showed, in our terminology, that the roots hierarchy is proper. We present a proof of:

$7^{1/3}$ *does not satisfy any quadratic equation over the integers.*

which contains most of their ideas. Our presentation is suitable for an undergraduate class (and was presented there).

Assume, by way of contradiction, that there exists $a_2, a_1, a_0 \in \mathsf{Z}$ such that

$$a_2 \times 7^{2/3} + a_1 \times 7^{1/3} + a_0 \times 1 = 0$$

We can assume that the gcd of a_2, a_1, a_0 is 1 since otherwise we could divide the gcd out. In particular 7 does not divide all three of a_2, a_1, a_0.

Multiply the equation by $1, 7^{1/3}, 7^{2/3}$ to get

$$\begin{aligned}
a_2 \times 7^{2/3} &\quad + a_1 \times 7^{1/3} &\quad + a_0 \times 1 &\quad = 0 \\
a_1 \times 7^{2/3} &\quad + a_0 \times 7^{1/3} &\quad + 7a_2 \times 1 &\quad = 0 \\
a_0 \times 7^{2/3} &\quad + 7a_2 \times 7^{1/3} &\quad + 7a_1 \times 1 &\quad = 0
\end{aligned}$$

We rewrite this as a matrix times a vector equals the 0 vector:

Let

$$A = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 7a_2 \\ a_0 & 7a_2 & 7a_1 \end{pmatrix}$$

and

$$v = \begin{pmatrix} 7^{2/3} \\ 7^{1/3} \\ 1 \end{pmatrix}$$

and

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$Av = \vec{0}$ and $v \neq \vec{0}$, so A must have det 0, so $A \pmod 7$ must have det 0:

$$A \pmod 7 = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 0 \\ a_0 & 0 & 0 \end{pmatrix}$$

If we expand this matrix on the last row we get that the det is a_0^3. Hence $a_0^3 \equiv 0 \pmod 7$, so $a_0 \equiv 0 \pmod 7$. Let $a_0 = 7b_0$. Now

$$A = \begin{pmatrix} a_2 & a_1 & 7b_0 \\ a_1 & 7b_0 & 7a_2 \\ 7b_0 & 7a_2 & 7a_1 \end{pmatrix}$$

Since A has det 0, so does A with the last col divided by 7. Hence the following matrix, B, had det 0:

$$B = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 7b_0 & a_2 \\ 7b_0 & 7a_2 & a_1 \end{pmatrix}$$

Since B has det 0, so does $B \pmod 7$.

$$B \pmod 7 = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 0 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$$

Expanding on the last row we get that the det of $B \pmod 7$ is $-a_1^3$. If $-a_1^3 \equiv 0 \pmod 7$ then $a_1 \equiv 0 \pmod 7$. Let $a_1 = 7b_1$. Hence

$$B = \begin{pmatrix} a_2 & 7b_1 & b_0 \\ 7b_1 & 7b_0 & a_2 \\ 7b_0 & 7a_2 & 7b_1 \end{pmatrix}$$

Since B has det 0, so does C which we obtain by dividing every element of the middle col by 7:

$$C = \begin{pmatrix} a_2 & b_1 & b_0 \\ 7b_1 & b_0 & a_2 \\ 7b_0 & a_2 & 7b_1 \end{pmatrix}$$

Since C has det 0, so does $C \pmod 7$ which is:

$$C \pmod 7 = \begin{pmatrix} a_2 & b_1 & b_0 \\ 0 & b_0 & a_2 \\ 0 & a_2 & 0 \end{pmatrix}$$

The det of this by expansion on bottom row is a_2^3. Since $a_2^3 \equiv 0 \pmod 7$, $a_2 \equiv 0 \pmod 7$.

So we have a_0, a_1, a_2 are all divisible by 7. This contradicts a_0, a_1, a_2 being in lowest terms.