The Roots Hierarchy
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1 Introduction

The main proof in this note is from Problems from the Book by Dospinescu and Andreescu.

We want to classify real numbers in terms of their complexity.

Def 1.1 Let $d \in \mathbb{N}$.

1. $\mathbb{Z}_d[x]$ is the set of polynomials of degree $d$ over $\mathbb{Z}$ (the integers).

2. roots$_d$ is the set of roots of polynomials in $\mathbb{Z}_d[x]$. Note that roots$_1 = \mathbb{Q}$.

Clearly

$$\text{roots}_1 \subseteq \text{roots}_2 \subseteq \text{roots}_3 \subseteq \cdots$$

We want to show that

$$\text{roots}_1 \subset \text{roots}_2 \subset \text{roots}_3 \subset \cdots$$

2 $7^{1/3} \notin \text{roots}_2$ 

In Problems from the Book by Dospinescu and Andreescu they show that, for all $d$, $2^{1/d}$ does not satisfy any $d - 1$-degree polynomial over the rationals. Hence they showed, in our terminology, that the roots hierarchy is proper. We present a proof of:

$7^{1/3}$ does not satisfy any quadratic equation over the integers.

which contains most of their ideas. Our presentation is suitable for an undergraduate class (and was presented there).

Assume, by way of contradiction, that there exists $a_2, a_1, a_0 \in \mathbb{Z}$ such that
\( a_2 \times 7^{2/3} + a_1 \times 7^{1/3} + a_0 \times 1 = 0 \)

We can assume that the gcd of \( a_2, a_1, a_0 \) is 1 since otherwise we could divide the gcd out. In particular 7 does not divide all three of \( a_2, a_1, a_0 \).

Multiply the equation by \( 1, 7^{1/3}, 7^{2/3} \) to get

\[
\begin{align*}
\quad a_2 \times 7^{2/3} + a_1 \times 7^{1/3} + a_0 \times 1 &= 0 \\
\quad a_1 \times 7^{2/3} + a_0 \times 7^{1/3} + 7a_2 \times 1 &= 0 \\
\quad a_0 \times 7^{2/3} + 7a_2 \times 7^{1/3} + 7a_1 \times 1 &= 0 
\end{align*}
\]

We rewrite this as a matrix times a vector equals the 0 vector:

Let

\[
A = \begin{pmatrix}
a_2 & a_1 & a_0 \\
a_1 & a_0 & 7a_2 \\
a_0 & 7a_2 & 7a_1
\end{pmatrix}
\]

and

\[
v = \begin{pmatrix}
7^{2/3} \\
7^{1/3} \\
1
\end{pmatrix}
\]

and

\[
\vec{0} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\( Av = \vec{0} \) and \( v \neq \vec{0} \), so \( A \) must have det 0, so \( A \) (mod 7) must have det 0:
\[
A \pmod{7} = \begin{pmatrix}
    a_2 & a_1 & a_0 \\
    a_1 & a_0 & 0 \\
    a_0 & 0 & 0
\end{pmatrix}
\]

If we expand this matrix on the last row we get that the det is \(a_0^3\). Hence \(a_0^3 \equiv 0 \pmod{7}\), so \(a_0 \equiv 0 \pmod{7}\). Let \(a_0 = 7b_0\). Now

\[
A = \begin{pmatrix}
    a_2 & a_1 & 7b_0 \\
    a_1 & 7b_0 & 7a_2 \\
    7b_0 & 7a_2 & 7a_1
\end{pmatrix}
\]

Since \(A\) has det 0, so does \(A\) with the last col divided by 7. Hence the following matrix, \(B\), had det 0:

\[
B = \begin{pmatrix}
    a_2 & a_1 & b_0 \\
    a_1 & 7b_0 & a_2 \\
    7b_0 & 7a_2 & a_1
\end{pmatrix}
\]

Since \(B\) has det 0, so does \(B \pmod{7}\).

\[
B \pmod{7} = \begin{pmatrix}
    a_2 & a_1 & b_0 \\
    a_1 & 0 & a_2 \\
    0 & 0 & a_1
\end{pmatrix}
\]

Expanding on the last row we get that the det of \(B \pmod{7}\) is \(-a_1^3\). If \(-a_1^3 \equiv 0 \pmod{7}\) then \(a_1 \equiv 0 \pmod{7}\). Let \(a_1 = 7b_1\). Hence
\[
B = \begin{pmatrix}
a_2 & 7b_1 & b_0 \\
7b_1 & 7b_0 & a_2 \\
7b_0 & 7a_2 & 7b_1
\end{pmatrix}
\]

Since \( B \) has det 0, so does \( C \) which we obtain by dividing every element of the middle col by 7:

\[
C = \begin{pmatrix}
a_2 & b_1 & b_0 \\
7b_1 & b_0 & a_2 \\
7b_0 & a_2 & 7b_1
\end{pmatrix}
\]

Since \( C \) has det 0, so does \( C \mod 7 \) which is:

\[
C \mod 7 = \begin{pmatrix}
a_2 & 0 & b_0 \\
0 & b_0 & a_2 \\
0 & a_2 & 0
\end{pmatrix}
\]

The det of this by expansion on bottom row is \( a_2^3 \). Since \( a_2^3 \equiv 0 \mod 7 \), \( a_2 \equiv 0 \mod 7 \).

So we have \( a_0, a_1, a_2 \) are all divisible by 7. This contradicts \( a_0, a_1, a_2 \) being in lowest terms.