Seeking an Easier Proof of a Weaker Result In Multiparty Comm Comp

by William Gasarch

January 21, 2022

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2. They want to know if $a_1 + \cdots + a_k = 2^{n+1} - 1$.
3. **Easy Solution** A_1 says a_2, A_2 then computes sum and then says YES if sum is $2^{n+1} - 1$, NO if not.
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Let $\mathrm{MPCC}(k, n)$ be the multiparty comm complexity of this problem. k is constant.

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Notation $\chi(k, N)$ is the min number of colors needed to color $\{1, \ldots, N\}^k$ such that there are no monochromatic isosceles L's.

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So we are done! the answer is $\lg(\chi(k, 2^n)$. Or are we?

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$$\omega(1) \le \lg(\chi(k-1, 2^n)) \text{ or } \omega(1) \le \mathrm{MPCC}(k, n)$$

PRO They used this for lower bounds on branching programs.

Further Results

Beigel-Gasarch-Glenn (2006)
`https://www.cs.umd.edu/~gasarch/BLOGBOOK/foreheadserious.pdf`

1. $\Omega(\log\log n) \le \mathrm{MPCC}(3, n)$.
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1) The answer is probably the upper bound.
2) Proving this will be difficult. Oh well.

A Different Open Question

Is there an easy proof that $\mathrm{MPCC}(3, n) < n$?

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I posted on this and Dean Foster responded with a proof that

$$\mathrm{MPCC}(3, n) < \frac{n}{2} + O(1).$$

$\mathrm{MPCC}(3, n) \le \frac{n}{2} + O(1)$

Foster: `https://www.cs.umd.edu/~gasarch/BLOGBOOK/DONE/foreheadfun.pdf`

1. A:$a_0 \cdots a_{n-1}$, B:$b_0 \cdots b_{n-1}$, C:$c_0 \cdots c_{n-1}$.
2. A says: $c_0 \oplus b_{n/2}, \ \cdots \ , \ c_{n/2-1} \oplus b_{n-1}$.
3. Bob knows c_i's so he now knows $b_{n/2}, \ldots, b_{n-1}$.
   Bob knows a_i's and c_i's so he can compute
   $a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$
   $s = 1^{n/2}$: Bob says (MAYBE,z). $s \ne 1^{n/2}$: Bob says NO.
4. Carol knows b_i's so she now knows $c_0, \ldots, c_{n/2-1}$.

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4. Carol knows b_i's so she now knows $c_0, \ldots, c_{n/2-1}$.
   Carol knows the carry bit z so she can compute
   $a_0 \cdots a_{n/2} + b_0 \cdots b_{n/2} + c_0 \cdots c_{n/2} + z = t$
   $t = 1^{n/2}$: Carol says YES. $t \ne 1^{n/2}$: Carol says NO.

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Can extend to get $\mathrm{MPCC}(k, n) \le \frac{n}{k-1} + O(1)$.

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Caveat I have not defined **easy** rigorously.