Truth Tellers and Normals
Exposition by William Gasarch

1 Introduction

All of the ideas here are from *On a Logical Problem* by Pavel Blecher. He dealt with the problem of N people in a room of which over half are truth tellers. We deal with t truth tellers, n normals, where $t > n$. He provides matching upper and lower bounds. We just give an upper bound; however, we suspect that his methods can be used to obtain a matching lower bound.

Def 1.1 A *truth teller* always tells the truth. A *normal*, given a question, will answer it either honestly or not.

Here is the question we consider:

You are in a room with n normals and t truthtellers with $t > n$. (If $t \le n$ then it is impossible to find out everyone's status- I leave that to the reader.) They all know who each other is but you do not. You may ask them YES-NO questions. You want to find out everyone's status. What is the least number of questions you need to ask? Note that the normals will conspire to make you ask as many questions as possible.

2 Notation and Needed Lemmas

Def 2.1 Let $n, t \in \mathbb{N}$ with $t > n$. *You are in a room with n normals and t truthtellers.* $f(n, t)$ is the least number of questions you need to determine everyone's status.

The more truth tellers the better off you are. We state the following lemma and leave the proof to the reader.

Lemma 2.2 *Let $n, t \in \mathbb{N}$ with $t > n$. You are in a room with n normals and t truthtellers. If $n' + t' = n + t$ and $n' \le n$, $t \ge t'$ then $f(n', t') \le f(n, t)$.*

Lemma 2.3 *Let $n, t \in \mathbb{N}$ with $t > n$. You are in a room with n normals and t truthtellers. If $n + 1$ people all agree on a question Q then what they all say is the correct answer.*

Proof: Look at the $n + 1$ people. At least one is a truth teller. Hence what he says is correct. And they all agree with him. ∎

Lemma 2.4 *Let $n, t \in \mathbb{N}$ with $t > n$. You are in a room with n normals and t truthtellers. Let P be one of the people. We ask q people $P_1, \ldots, P_q$ Is P a normal?. L of them say YES and $L - 1$ of them say NO (so $q = 2L - 1$, odd). Then of the people in $\{P_1, \ldots, P_q, P\}$ at least $L = \frac{q+1}{2}$ (so at least half) are normals.*

Proof:

If P is a normal then the $L-1$ that say P *is a truth teller* must be normals. Hence there are L normals: P and the $L-1$ who lied about P.

If P is a truth teller then the L that say P *is a normal* must be normals. Hence there are L normals: the L who lied about P. ▮

3 An Algorithm and its Complexity

Theorem 3.1 *Let $n, t \in \mathbb{N}$ with $t > n$. You are in a room with n normals and t truthtellers. $f(n,t) \le 2n + t - 1$.*

Proof: We use the following algorithm.

The people are $P_1, \ldots, P_{n+t}$.

Let Q be the question

Is P_{n+t} normal?

ALGORITHM

If $t = 1$ then $n = 0$ so output that the one person is a truth teller. This took $0 \le 2n + t - 1 = 0$ questions. Henceforth assume $t \ge 2$.

Ask $P_1, P_2, \ldots$ the question Q until one of the following occurs.

1. After q questions $n+1$ people all say NO- that is they all say that P_{n+t} is a truth teller. By Lemma 2.3 P_{n+t} is a truth teller. (We do not know the status of those who said he was a truth teller- they were honest on that question but we do not not know if they are always honest.) There are $q-n-1$ people who said P_{n+t} is a normal. They lied so they must all be normals.

   RECAP: We have asked q questions, found out the status of $q-n$ people (the truth teller and the $q-n-1$ normals), we have found a truth teller, and there are $(n+t)-(q-n) = 2n+t-q$ people left whose status we do not know.

   FINISH UP: Ask the truth teller P_{n+t} about those people to find their status. This will take $2n+t-q-1$ questions (we don't need to ask about the last person's status by process of elimination).

   HOW MANY QUESTIONS: We initially asked q questions and then $2n+t-q-1$ questions, so a total of $2n+t-1$ questions.

2. After q questions there are L who say P_{n+t} *is a normal* and $L-1$ who say P_{n+t} *is a truth teller*. Note that $q = 2L-1$. By Lemma 2.4, of $\{P_1, \ldots, P_q, P_{n+t}\}$ at least $\frac{q+1}{2}$ are normals. Remove all of $\{P_1, \ldots, P_q, P_{n+t}\}$ temporarily. KEY: We've removed either the same number of normals as truth tellers or MORE normals than truthtellers so we will still have left MORE truth tellers than normals and can recurse.

   RECAP: We have asked $q = 2L-1$ questions. There is a set of $q+1$ people who we have removed. We do not know their status; however, once we know the status of P_{n+t} we will know that which L of them are normals.

FINISH UP: We have temporarily removed them. There are $(n+t)-(q+1)$ people left. Of these there are at most $n-\frac{q+1}{2}$ normals and at least $t-\frac{q+1}{2}$ truthtellers. We now solve that problem recursively, and when we are done ask a truth teller about P_{n+t} which will determine the status of those we removed.

HOW MANY QUERIES: Clearly:

$$f(n,t) \le q+1+f\left(n-\frac{q+1}{2}, t-\frac{q+1}{2}\right).$$

Inductively this is

$$\le q+1+2\left(n-\frac{q+1}{2}\right)+\left(t-\frac{q+1}{2}\right)-1$$

$$= q+1+2n-(q+1)+t-\frac{q+1}{2}-1 = 2n+t-\frac{q+1}{2}-1 \le 2n+t-1.$$

We need to show that one of the two cases occurs. After $2n+1$ questions either:

1. $\ge n+1$ people said P_{n+t} *is a truth teller*. Hence there is a point where the number of people saying *truth teller* is $n+1$. At the earliest such point Case 1 would occur.

2. $\le n$ people said P_{n+t} *is a truth teller*. Then $\ge n+1$ people said P_{n+t} *is a normal*. Hence there is a point where the number of people saying *normal* is greater than the number of people saying *truth teller*. At the earliest such point Case 2 would occur.

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