**Placing Pennies on a Chessboard to Obtain Distinct Distances**
**By William Gasarch**

# 1 Introduction

Neither the problem nor the solution of the topic of this paper are mine. I heard the problem at the Boston AMS-MAA joint meeting in Boston in 2023, though it seems to be due to Matt Parker (I have not been able to find a reference). I blogged about it asking for a solution and was pointed to a website which had a solution. This paper is essentially that solution, only (1) with details that are normally (and correctly) left to the reader filled in, (2) some suggested open problems, and (3) references to papers with similar problems.

**Convention 1.1** Throughout this paper (1) an $n \times n$ chessboard has all squares $1 \times 1$, (2) pennies have diameter 1, (3) the distance between two pennies is the distance between their centers.

**Def 1.2** Let $k, n \in \mathbb{N}^+$ with $k \leq n$.

1. $(k, n)$ is *placeable* if there is a way to place $k$ pennies on an $n \times n$ chessboard such that all the distances are distinct.
2. $n$ is *placeable* if $(n, n)$ is placeable.

Matt Parker asked the following:

*Which $n$ are placeable?*

Oscar Cunningham, in his blog here:
`https://oscarcunningham.com/670/unique-distancing-problem/`
solved the problem with help from Gal Holowitz. We state what he claims and give comments on it:

1. Oscar Cunningham used a computer search and found that, for $3 \leq n \leq 7$, $n$ is placeable. This was also shown by Erdős and Guy [1] (see Section 4 for more on that paper) who we suspect did not use a computer. We give the placements of Erdős and Guy in the appendix.
2. Oscar Cunningham used a computer search to show that, for $8 \leq n \leq 11$, $n$ is not placeable. We pose the following open question: find a human-readable proof.

3. Gal Holowitz used a computer search to show that, for $12 \le n \le 14$, $n$ is no placeable. We pose the following open question: find a human-readable proof.

4. Oscar Cunningham states that

   *But for the $n = 15$ case their code is still running!*

   That was written in 2020 so one presumes the code has stopped running. In any case, the $n = 15$ case did not seem amenable to a computer search. Oscar Cunningham then gave an elegant human-readable proof that 15 is not placeable. We present that proof in this paper.

5. Oscar Cunningham says that for all $n \ge 16$

   > *it is trivial that there are no solutions (what we call placements) because the number of pairs of counters (what we call pennies) is greater than the number of possible distances on the grid.*

   While this is true, we did not think it was trivial. Hence we have provided a short proof.

# 2 15 is NOT Placeable

**Def 2.1** Let $f(n)$ be the number of numbers that can be written as the sum of 2 squares in at least 2 ways using numbers from $\{0, \ldots, n-1\}$.

**Lemma 2.2** *The number of distances between squares on the $n \times n$ chessboard is*

$$\le \frac{n(n-1)}{2} + n - 1 - f(n).$$

**Proof:** The set of distances between squares is the set of distances from the left bottom square (LBS) to all of the other squares, minus repeats. First look at the distance from the LBS to the top right square. Then from the LBS to the two squares that are furthest away in the second-to-top row. Etc. This is $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$. Then we add in all of the squares in the bottom row except the LBS. That's $n - 1$. We then subtract the number of repeats which is $\ge f(n)$. Hence the number of distances is

$$\le 1 + 2 + 3 + \cdots + (n-1) + (n-1) - f(n) = \frac{n(n-1)}{2} + n - 1 - f(n)$$

∎

**Theorem 2.3** *15 is not placeable.*

**Proof:**

$f(15) \geq 14$. We list 14 numbers that can be written as $a^2 + b^2$ with $a, b \in \{1, \ldots, 15\}$. It turns out that $f(15) = 14$ though we do not need that and do not prove it.

$25 = 0 \times 0 + 5 \times 5 = 3 \times 3 + 4 \times 4$
$50 = 1 \times 1 + 7 \times 7 = 5 \times 5 + 5 \times 5$
$65 = 1 \times 1 + 8 \times 8 = 4 \times 4 + 7 \times 7$
$85 = 2 \times 2 + 9 \times 9 = 6 \times 6 + 7 \times 7$
$100 = 0 \times 0 + 10 \times 10 = 6 \times 6 + 8 \times 8$
$125 = 2 \times 2 + 11 \times 11 = 5 \times 5 + 10 \times 10$
$130 = 3 \times 3 + 11 \times 11 = 7 \times 7 + 9 \times 9$
$145 = 1 \times 1 + 12 \times 12 = 8 \times 8 + 9 \times 9$
$169 = 0 \times 0 + 13 \times 13 = 5 \times 5 + 12 \times 12$
$170 = 1 \times 1 + 13 \times 13 = 7 \times 7 + 11 \times 11$
$185 = 4 \times 4 + 13 \times 13 = 8 \times 8 + 11 \times 11$
$200 = 10 \times 10 + 10 \times 10 = 2 \times 2 + 14 \times 14$
$205 = 3 \times 3 + 14 \times 14 = 6 \times 6 + 13 \times 13$
$221 = 10 \times 10 + 11 \times 11 = 5 \times 5 + 14 \times 14$

Since $f(15) \geq 14$, by Lemma 2.2 the number of possible distances is

$$\leq \frac{15 \times 14}{2} + 14 - 14 = \frac{15 \times 14}{2} = \binom{15}{2}.$$

AH-HA! To place 15 pennies you need to achieve $\binom{15}{2}$ distances. Hence EVERY distance must appear.

All of the distances are of the form $\sqrt{a^2 + b^2}$ where $1 \leq a \leq b \leq 14$. We list here the top 6 distances.

1. $\sqrt{14^2 + 14^2} = \sqrt{392}$.
2. $\sqrt{13^2 + 14^2} = \sqrt{365}$.
3. $\sqrt{12^2 + 14^2} = \sqrt{340}$.
4. $\sqrt{13^2 + 13^2} = \sqrt{338}$.
5. $\sqrt{11^2 + 14^2} = \sqrt{317}$.
6. $\sqrt{12^2 + 13^2} = \sqrt{313}$.

We try to place pennies to get these distances and show that we cannot.

The largest distance, $\sqrt{392}$, can only be achieved by having two pennies in opposite diagonal corners. Hence we can assume that $p_1$ and $p_2$ are as in Figure 1. We use $X$ to indicate spots where no penny can go since they are equidistant from $p_1$ and $p_2$.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | | | | | | | | | | | | | | $X$ |
| | | | | | | | | | | | | | $X$ | |
| | | | | | | | | | | | | $X$ | | |
| | | | | | | | | | | | $X$ | | | |
| | | | | | | | | | | $X$ | | | | |
| | | | | | | | | | $X$ | | | | | |
| | | | | | | | | $X$ | | | | | | |
| | | | | | | | $X$ | | | | | | | |
| | | | | | | $X$ | | | | | | | | |
| | | | | | $X$ | | | | | | | | | |
| | | | | $X$ | | | | | | | | | | |
| | | | $X$ | | | | | | | | | | | |
| | | $X$ | | | | | | | | | | | | |
| | $X$ | | | | | | | | | | | | | |
| $X$ | | | | | | | | | | | | | | $p_2$ |

Figure 1: Placement of $p_1$ and $p_2$

1. $d(p_1, p_2) = \sqrt{392}$

The second largest distance, $\sqrt{13^2 + 14^2} = \sqrt{365}$ can only be achieved by going from corner to the space next to the oppoiste diagonal corner . Given where we place $p_1$ and $p_2$ the only way to achieve this is to place $p_3$ as in Figure 2. We once again place $X$'s in places where no penny can go.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_3$ | $X$ | | | | | | | | | | | | $X$ |
| $X$ | $X$ | | | | | | | | | | | | $X$ | |
| | | | | | | | | | | | | $X$ | | |
| | | | | | | | | | | | $X$ | | | |
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| | | | | | | | $X$ | | | | | | | |
| | | | | | | $X$ | | | | | | | | |
| | | | | | $X$ | | | | | | | | | |
| | | | | $X$ | | | | | | | | | | |
| | | | $X$ | | | | | | | | | | | |
| | | $X$ | | | | | | | | | | | | |
| | $X$ | | | | | | | | | | | | | $X$ |
| $X$ | | | | | | | | | | | | | $X$ | $p_2$ |

Figure 2: Placement of $p_1, p_2, p_3$

1. $d(p_1, p_2) = \sqrt{392}$
2. $d(p_1, p_3) = 1$,
3. $d(p_2, p_3) = \sqrt{365}$

The third largest distance, $\sqrt{12^2 + 14^2} = \sqrt{340}$ can only be acheived by going from a corner to two away from the diagonally opposite corner (pennies $p_2$ and $p_5$ in Figure 4). Given how we placed $p_1, p_2, p_3$ there are 3 ways to place a penny to achieve this. This is not what we want! We want there to be only one way! Hence we put off getting this distance for now.

The fourth largest distance, $\sqrt{13^2 + 13^2} = \sqrt{338}$, is from a corner to the square that is diagonally next to the diagonally opposite corner. Given where we placed $p_1, p_2, p_3$, the only way to achieve this is placing $p_4$ as in Figure 3

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_3$ | $X$ | | | | | | | | | | | $X$ | $X$ |
| $X$ | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | |
| | $X$ | | | | | | | | | | $X$ | $X$ | | |
| | | | | | | | | | | $X$ | $X$ | | | |
| | | | | | | | | | $X$ | $X$ | | | | |
| | | | | | | | | $X$ | $X$ | | | | | |
| | | | | | | | $X$ | $X$ | | | | | | |
| | | | | | | $X$ | $X$ | | | | | | | |
| | | | | | $X$ | $X$ | | | | | | | | |
| | | | | $X$ | $X$ | | | | | | | | | |
| | | | $X$ | $X$ | | | | | | | | | | |
| | | $X$ | $X$ | | | | | | | | | | | |
| | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | $X$ |
| $X$ | $X$ | | | | | | | | | | | $X$ | $p_4$ | $X$ |
| $X$ | | | | | | | | | | | | $X$ | $X$ | $p_2$ |

Figure 3: Placement of $p_1, p_2, p_3, p_4$

1. $d(p_1, p_2) = \sqrt{392}$
2. $d(p_1, p_3) = 1$
3. $d(p_1, p_4) = \sqrt{338}$
4. $d(p_2, p_3) = \sqrt{365}$
5. $d(p_2, p_4) = \sqrt{2}$
6. $d(p_3, p_4) = \sqrt{313}$

We now place the third largest distance, $\sqrt{14^2+12^2}=\sqrt{340}$. As noted earlier, this can only be acheived by going from a corner to two away from the diagonally opposite corner Given how we place $p_1, p_2, p_3, p_4$ there we are forced to place $p_5$ as in Figure 4.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_3$ | $X$ | | | | | | | | | | | $X$ | $X$ |
| $X$ | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | |
| $p_5$ | $X$ | | | | | | | | | | $X$ | $X$ | | |
| | | | | | | | | | | $X$ | $X$ | | | |
| | | | | | | | | | $X$ | $X$ | | | | |
| | | | | | | | | $X$ | $X$ | | | | | |
| | | | | | | | $X$ | $X$ | | | | | | |
| | | | | | | $X$ | $X$ | | | | | | | |
| | | | | | $X$ | $X$ | | | | | | | | |
| | | | | $X$ | $X$ | | | | | | | | | |
| | | | $X$ | $X$ | | | | | | | | | | |
| | | $X$ | $X$ | | | | | | | | | | | |
| | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | $X$ |
| $X$ | $X$ | | | | | | | | | | | $X$ | $p_4$ | $X$ |
| $X$ | | | | | | | | | | | | $X$ | $X$ | $p_2$ |

Figure 4: Placement of $p_1, p_2, p_3, p_4, p_5$

1. $d(p_1,p_2)=\sqrt{392}$
2. $d(p_1,p_3)=1$
3. $d(p_1,p_4)=\sqrt{338}$
4. $d(p_1,p_5)=2$
5. $d(p_2,p_3)=\sqrt{365}$
6. $d(p_2,p_4)=\sqrt{2}$
7. $d(p_2,p_5)=\sqrt{340}$
8. $d(p_3,p_4)=\sqrt{313}$
9. $d(p_3,p_5)=\sqrt{5}$
10. $d(p_4,p_5)=\sqrt{290}$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_3$ | $X$ | $p_2^6$ | | | | | | | | | | $X$ | $X$ |
| $X$ | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | |
| $p_5$ | $X$ | | | | | | | | | | $X$ | $X$ | | |
| $p_6^2$ | | | | | | | | | | $X$ | $X$ | | | |
| | | | | | | | | | $X$ | $X$ | | | | |
| | | | | | | | | $X$ | $X$ | | | | | |
| | | | | | | | $X$ | $X$ | | | | | | |
| | | | | | | $X$ | $X$ | | | | | | | |
| | | | | | $X$ | $X$ | | | | | | | | |
| | | | | $X$ | $X$ | | | | | | | | | |
| | | | $X$ | $X$ | | | | | | | | | | |
| | | $X$ | $X$ | | | | | | | | | | | $p_6^1$ |
| | $X$ | $X$ | | | | | | | | | | $X$ | $X$ | $X$ |
| $X$ | $X$ | | | | | | | | | | | $X$ | $p_4$ | $X p_6^5$ |
| $X$ | | | | | | | | | | | $p_6^1$ | $X p_6^3$ | $X$ | $p_2$ |

Figure 5: Attempt to Place $p_6$

We now try to place $p_6$ to create the fifth largest distance, $\sqrt{11^2 + 14^2} = \sqrt{317}$. We show this cannot be done. For $1 \le i \le 5$ we place $p_6$ in the place(s) it needs to be to get $d(p_i, p_6) = \sqrt{317}$. We label those places $p_6^i$.

Recall that we cannot use a distance twice.

1. For both $p_6^1$'s, $d(p_6^1, p_4) = \sqrt{5} = d(p_3, p_5)$.

2. For one of the $p_6^2$, $d(p_6^2, p_5) = d(p_1, p_3) = 1$. For the other one $d(p_6^2, p_3) = d(p_1, p_5) = 2$.

3. $p_6^3$ is on an $X$ spot.

4. There is no $p_6^4$ since it would be off the board.

5. $p_6^5$ is on an $X$ spot.

∎

# 3 For all $n \ge 16$, $n$ is not placeable

**Theorem 3.1**

1. *Let* $a, b, c, d \in \mathbb{N}$ *with* $(a, b) \neq (c, d)$ *such that* $4 \leq a, c$ *and* $1 \leq b, d \leq 3$. *Then* $a^2 + b^2 \neq c^2 + d^2$, *so* $5(a^2 + b^2) \neq 5(c^2 + d^2)$.
2. *If* $n \geq 33$ *then* $f(n) \geq n$.
3. *If* $n \geq 16$ *then* $f(n) \geq n$.
4. *For all* $n \geq 16$ *the number of distances between spaces on the* $n \times n$ *chess board is* $\leq \binom{n}{2} - 1$.
5. *For all* $n \geq 16$, $n$ *is not placeable.*

**Proof:**
1) There are three cases.
**Case 1:** $a = c$. Then we have $b \neq d$. We can assume $b^2 < d^2$. Hence we get

$$a^2 + b^2 < c^2 + d.$$

**Case 2:** $a < c$ so $c - a > 0$. Since $4 \leq a, c$ and $a < c$ we have $a + c \geq 9$. Using all of this we get:

$$c^2 - a^2 = (c + a)(c - a) \geq a + c \geq 9$$

so

$$a^2 + 9 \leq c^2.$$

Now we look at $a^2 + b^2$. Since $b \leq 3$, $b^2 \leq 9$. Hence

$$a^2 + b^2 \leq a^2 + 9 \leq c^2.$$

Since $d \geq 1$, $c^2 < c^2 + d^2$. Combining this with the above equation we get

$$a^2 + b^2 \leq a^2 + 9 \leq c^2 < c^2 + d^2.$$

Hence $a^2 + b^2 \neq c^2 + d^2$.
**Case 3:** $c < a$. Similar to Case 2.

2) For each $(a, b)$ with $a > b > 0$ we can write $5(a^2 + b^2)$ as a sum of two squares in two different ways:

$$5(a^2 + b^2) = (|a - 2b|)^2 + (2a + b)^2 = (a + 2b)^2 + (2a - b)^2.$$

Since $a > b > 0$, we see that $2a+b$ is larger than the other numbers, so these representations are distinct.

For $n \geq 33$ we will use this to get many elements of $\{1, \ldots, n\}$ that can be written as the sum of two squares in two different ways. We need to spilt up cases for $n$ even and $n$ odd. Note that we will only need $2a + b \leq n$ since that is the largest number.

**Case 0:** $n$ **even:** Let $4 \leq a \leq \frac{n-4}{2}$ and $1 \leq b \leq 3$. There are $\frac{n-10}{2} \times 3 - 2 = \frac{3n}{2} - 17$ pairs $(a, b)$. When $n \geq 34$, this is $\geq n$.

We now show that $2a + b \leq n$.

$$2a + b \leq 2\left(\frac{n-4}{2}\right) + 3 < n$$

**Case 1:** $n$ **odd:** Let $4 \leq a \leq \frac{n-3}{2}$ and $1 \leq b \leq 3$. We have $\frac{n-9}{2} \times 3 - 2 = \frac{3n}{2} - \frac{31}{2}$ pairs. When $n \geq 31$, this is $\geq n$.

We now show that $2a + b \leq n$.

$$2a + b \leq 2\left(\frac{n-3}{2}\right) + 3 = n.$$

3) Part 2 we only need to prove the theorem for $n = 16, \ldots, 32$.

By the proof of Theorem 2.3 we know that $f(15) \geq 14$. We indicate what numbers to add to that list of 14.

16: Add 225 and 250:

$225 = 0 \times 0 + 15 \times 15 = 9 \times 9 + 12 \times 12$

$250 = 5 \times 5 + 15 \times 15 = 9 \times 9 + 13 \times 13$

So $f(16) \geq 14 + 2 = 16$.

17,18: Add 260 and 265:

$260 = 2 \times 2 + 16 \times 16 = 8 \times 8 + 14 \times 14$

$265 = 11 \times 11 + 12 \times 12 = 3 \times 3 + 16 \times 16$

So $f(17) \geq 16 + 2 = 18$.

Hence $f(18) \geq 18$.

19,20,21,22: Add 365, 370, 377, 410:

$365 = 13 \times 13 + 14 \times 14 = 2 \times 2 + 19 \times 19$

$370 = 3 \times 3 + 19 \times 19 = 9 \times 9 + 17 \times 17$

$377 = 11 \times 11 + 16 \times 16 = 4 \times 4 + 19 \times 19$

$410 = 11 \times 11 + 17 \times 17 = 7 \times 7 + 19 \times 19$

So $f(19) \geq 18 + 4 = 22$.
Hence $f(20), f(21), f(22)$ are all $\geq 22$.

23,24,25,26,27,28,29,30: Add 530, 533, 545, 565, 578, 610, 629, 650.

$530 = 13 \times 13 + 19 \times 19 = 1 \times 1 + 23 \times 23$
$533 = 2 \times 2 + 23 \times 23 = 7 \times 7 + 22 \times 22$
$545 = 16 \times 16 + 17 \times 17 = 4 \times 4 + 23 \times 23$
$565 = 6 \times 6 + 23 \times 23 = 9 \times 9 + 22 \times 22$
$578 = 17 \times 17 + 17 \times 17 = 7 \times 7 + 23 \times 23$
$610 = 13 \times 13 + 21 \times 21 = 9 \times 9 + 23 \times 23$
$629 = 10 \times 10 + 23 \times 23 = 2 \times 2 + 25 \times 25$
$650 = 11 \times 11 + 23 \times 23 = 17 \times 17 + 19 \times 19$
So $f(23) \geq 22 + 8 = 30$.
Hence $f(23), f(24), f(25), f(26), f(27), f(28), f(29), f(30)$ are all $\geq 30$.

31,32: Add 962 and 965.

$962 = 11 \times 11 + 29 \times 29 = 1 \times 1 + 31 \times 31$
$965 = 17 \times 17 + 26 \times 26 = 2 \times 2 + 31 \times 31$
So $f(31) \geq 30 + 2 = 32$.
Hence $f(32) \geq 32$.

4) By Lemma 2.2 the number of distances is $\binom{n}{2} + n - 1 - f(n)$. By Part 3, $f(n) \geq n$.
Hence the number of distances is $\leq \binom{n}{2} - 1$.

5) Let $n \geq 16$. If $n$ is placeable then there is a way to place $n$ pennies on the $n \times n$ chessboard so that all $\binom{n}{2}$ distances occur. By Part 4 there are $\leq \binom{n}{2} - 1$ distances. Hence this cannot happen. ∎

# 4 Similar Problems

Erdős and Guy [1] posed the following question: Given $n$, what is the max $k$ such that $(k, n)$ is placeable. They showed that if $(k, n)$ is placeable and $n$ is large then, for all $\epsilon > 0$,

$$\Omega(n^{2/3-\epsilon}) \leq k \leq O\left(\frac{n}{(\log n)^{1/4}}\right).$$

The *Erdős Distance Problem* is the following: Given $n$ points in the plane what is the minimum number of distinct distances. This is denoted by $g(n)$. Erdős showed that

$$\Omega(\sqrt{n}) \le g(n) \le O\left(\frac{n}{\sqrt{\log n}}\right).$$

Some sources say Erdős conjectured $(\forall c < 1)[g(n) = \Omega(n^c)]$. Some sources say that Erdős conjectured $g(n) \ge \Omega(\frac{n}{\sqrt{\log n}})$. There was steady progress on better lower bounds (see the Wikipedia page on *Erdos Distinct Distance Problem*) with the best result being by Guth and Katz [2], who showed $g(n) \ge \Omega(\frac{n}{\log n})$. This solves the first conjecture but not the second.

# 5 Open Problems

1. The placements for 3,4,5,6,7 are ad-hoc. We would like a theorem from which these (or some of these) placements are corollaries.

2. The proofs that, for $8 \le n \le 14$, $n$ is not placeable was done by a computer. While we are confident that these proofs are valid, we would like to see a human-readable proof. Perhaps like the proof that 15 is not placeable. We suspect that (a) for $n = 14$ this is quite possible, though it may be a bit longer than we like, and (b) for $n = 8$ it will need new ideas.

3. What happens if you ask these problems in higher dimensions?

4. What happens with other metrics?

# 6 Acknowledments

We thank Larry Washingtion for help with Theorem 3.1 and Cheung-Yuan Lee for proof-reading and helpful comments.

# References

[1] P. Erdos and R. Guy. Distinct distances between lattice points. *Elemente Der Mathematik*, 25:121–123, 1970.
https://users.renyi.hu/~p_erdos/1970-03.pdf.

[2] L. Guth and N. H. Katz. On the Erdos distinct distances problem in the plane. *Annals of Mathematics*, 181:155–190, 2015.

# A Placements for 3,4,5,6,7

| | | |
|---|---|---|
| $p_1$ | | |
| | | |
| | $p_2$ | $p_3$ |

Figure 6: 3 is Placeable

| | | | |
|---|---|---|---|
| $p_1$ | | | |
| | | | |
| $p_2$ | | | $p_3$ |
| | | | $p_4$ |

Figure 7: 4 is Placeable

| | | | | |
|---|---|---|---|---|
| $p_1$ | | | | |
| | | | | |
| $p_2$ | | | $p_3$ | |
| | | | $p_4$ | |
| | | | | $p_5$ |

Figure 8: 5 is placeable

| | | | | | |
|---|---|---|---|---|---|
| | | | $p_4$ | | $p_5$ |
| | | | | | |
| | $p_3$ | | | | |
| | | | | | $p_6$ |
| $p_2$ | | | | | |
| $p_1$ | | | | | |

Figure 9: 6 is placeable

| | | | | | | |
|---|---|---|---|---|---|---|
| | | $p_4$ | | | | $p_7$ |
| | | | | | $p_6$ | |
| | | | | | | |
| | | | | | | |
| $p_2$ | $p_3$ | | | | | |
| | | | | | | |
| $p_1$ | | | $p_5$ | | | |

Figure 10: 7 is placeable