There is a long path in the divisor graph.

by

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Let D_n denote the graph whose vertices are the numbers $1, 2, \ldots, n$ with an edge joining a and b if either $a|b$ or $b|a$. Hegyvári has posed the problem of estimating the length of the longest simple path in the divisor graph. He and Erdös have shown that a simple path can have length at most $(1-\log 2)n$, for large n, see [1]. Recently Pomerance [2] has shown that the longest path has length which is $o(n)$. In this note we obtain in an estimate in the opposite direction. We exhibit a path of length

$$n \exp(-(2 + \varepsilon) \sqrt{\log n \operatorname{loglog} n}).$$

From now on, by a path, we shall always mean a simple path, i.e., a path which passes through a vertex at most once. The length of a path will be the number of vertices through which it passes.

The existence of this long path in D_n depends on finding a long path in the inclusion graph I_m, on subsets of $\{a_1, \ldots, a_m\}$. We shall need.

Lemma. If $k<m$ there is a path in the inclusion graph I_m which visits every k-element subset of $\{a_1, \ldots, a_m\}$ but which visits no subset with more than $k + 1$ elements.

Note: If $k = m - 1$, then the Gray code is such a path.

Proof. Let $M_1(a_1, \ldots, a_m)$ denote the path

$$a_1 \to a_1a_2 \to a_2 \to \ldots \to a_{m-1}a_m \to a_m.$$

Let $W_i(a_1, \ldots, a_m)$ denote the reversal of the path $M_i(a_1, \ldots, a_m)$. We define $M_i(a_1, \ldots, a_m)$ recursively:

$$M_i(a_1, \ldots, a_m) = a_1 \to a_1W_{i-1}(a_2, \ldots, a_m)$$
$$\to a_2 \to a_2W_{i-1}(a_3, \ldots, a_m)$$
$$\vdots$$
$$\to a_{m-i+1} \to a_{m-i+1}W_{i-1}(a_{m-i+2}, \ldots, a_m)$$

It is easily checked that the path $M_k(a_1, \ldots, a_m)$ satisfies the lemma.

We now prove:

Theorem. Given $\varepsilon > 0$ there is an n_o, so that , for all $n > n_o$ there is a simple path in D_n of length at least

$$n \exp(-(2 + \varepsilon) \sqrt{\log n \operatorname{loglog} n}).$$

Proof. Put $k + 1 = [\sqrt{\log n/\operatorname{loglog} n}]$. Let m be the greatest integer for which there is a prime $p_m < n^{1/k+1}$. Then, by the lemma, there is a simple path which visits $2\binom{m}{k}$ distinct subsets of $\{p_1, \ldots, p_m\}$ and no subsets with more than k + 1 elements. Therefore there is a path in D_n which visits $2\binom{m}{k}$ vertices.

Now $\binom{m}{k} \geq (\frac{m}{k})^k$ for $n > n_1$ and $m > (k + 1)n^{1/k+1}/ e\log n$ for $n > n_2$, by the prime number theorem.

With this choice of k we then have
$\binom{m}{k} > n \exp(-(2 + \varepsilon) \sqrt{\log n \log \log n})$ for all $n > n(\varepsilon)$.

References

[1] Erdös, Freud and Hegyvári. Arithmetical properties of permutations of integers. Acta Math. Acad. Sci. Hung (to appear).
[2] Pommerance. On the longest simple path in the divisor graph. Congressus Numerantium (to appear).

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