Longest simple path in the divisor graph, the $n = 1000$ case

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1 Introduction

After a challenge among our classmates which consisted to find the longest path in the divisor graph for the numbers between 1 and 1000, we found out that 666 is the best possible value. It was known that the optimal is 77 for the number between 1 and 100. We will prove that this chain is optimal.

Definition 1 *We denote D_n the n-divisorial graph, an undirected graph whit vertices $V = [1, n]$ and edges E such that $\{i, j\} \in E$ if and only if i is a multiple of j or the converse.*

Theorem 1 *The longest simple path in D_{1000} is of size 666.*

The path of size 666 is obtained with gurobi (see in appendix). The optimality is proved by studying the possible form of the optimal solution.

Definition 2 *For $i \geq 1$, we denote $f(i)$ the set of prime factor of i. For a finite sequence (u_i) of $\mathcal{N}$, we denote $F(u) = \bigcup_i f(u_i)$ and $M(u) = \max F(u)$.*

In what follow, we will consider a longest path $(u)_i$ in D_{1000} such that $M(u)$ is minimal.

Lemma 1 *For all prime number $p \leq M(u)$, $p \in F(u)$.*

The proof is by contradiction, if a prime number $p \leq M(u)$ is not a factor of an element of u, we could create a sequence u' by replacing all the occurrences of $M(u)$ in the prime decomposition of the element of u by p, u' would be a longest simple path in F_{1000} with $M(u') < M(u)$, which contradict the minimality of the longest path u for the function M.

Lemma 2 *We have $M(u) = 131$.*

The inequality $M(u) > 127$ (the prime number before 131) is obtained with gurobi which rapidly gives us a longest path of 665 in the subgraph of D_{1000} whose vertices are the numbers with prime factor at most 127, so as there is at least one path of size 666 we have to consider more prime numbers.

The inequality $M(u) \leq 131$ is more subtle:
In u we consider j blocks B_1, ..., B_j such that each block don't contains vertices less or equal to 14, and with each block only separated by vertices less or equal to 14. We clearly have $j \leq 15$ as there can be at most 14 separations between blocks. We can see that if a prime number $p > \frac{1000}{14}$ divide one element of a block, then it divide all of them, and that such a prime

number is unique for a block. Moreover we know that at east one block don't have a common prime factor $p > \frac{1000}{14}$ as a block with such common prime factor have at most 15 elements, so such a path would be of size at most $j * 15 + 14 \leq 239$, which we know is below optimal. This means that at most 14 primes numbers greater than 1000/14 appear as a factor of the elements of u. And 131 is the $14th$ such number, so $M(u) \leq 131$.

Another consequence of the last result is that there is only one block without prime factor greater than 1000/14, and that each block is separated by exactly one number between 1 and 14. So now in order to find the optimal, we can consider the equivalent problem: for every possible separation of the block without big prime factor $0 \leq k_1 < k_2 \leq 14$ (where $k_1 = 0$ means the block is at the start, the case where it is at the end don't have to be considered by symmetry), find the longest possible size for the block (so the longest path (h_i) from k_1 to k_2 in F_{1000} with $M_(h) < 1000/14$ and without using the vertices $1, ..., 14$, and then the longest path in the subgraph of F_{1000} on the vertices $1, ..., 14$ and the vertices v such that their greater prime factor is between $\frac{1000}{14}$ and 131 and where an edge between k_1 and k_2 is added to replace the missing block. This last edge have to be set as mandatory in the considered paths, as it will be replaced by the previously found block. And so the size of the longest possible chain will be the sum of the size of those two longest path, minus 2 if $k_1 > 0$ as k_1, k_2 are in both paths, minus 1 otherwise.

Gurobi take more time finding the best path for the 14 blocks with big prime factor (this path have a size around 130) than for the block with small prime factor (of size around 530). So in order to eliminate the k_1, k_2 with optimal solution strictly below 666 faster, we stop the search before finding the optimal path if we observe that 666 will not be obtain.

This method enable us to find solutions of size 666 and no more when we have

$$(k_1, k_2) \in \{(11, 12), (11, 13), (12, 13), (11, 14), (13, 14)\}$$

A A solution of size 666

The blocks with big prime factors are in blue. In this solution the block without big factor (in black) is between $(k_1, k_2) = (13, 14)$

711, 237, 474, 948, 316, 632, 158, 790, 395, 79, 869, 11, 979, 89, 712, 356, 178, 534, 267, 801, 3, 603, 201, 402, 804, 268, 536, 134, 670, 335, 67, 469, 938, 14, 854, 427, 61, 549, 183, 366, 732, 244, 976, 488, 122, 610, 305, 915, 15, 885, 295, 590, 118, 826, 413, 59, 531, 177, 354, 708, 236, 472, 944, 16, 848, 424, 212, 636, 318, 954, 477, 159, 795, 265, 530, 106, 742, 371, 53, 901, 17, 799, 47, 987, 329, 658, 94, 752, 376, 188, 564, 282, 846, 423, 141, 705, 235, 470, 940, 20, 860, 430, 215, 645, 129, 387, 774, 258, 516, 172, 344, 688, 86, 602, 301, 903, 43, 473, 946, 22, 506, 253, 759, 69, 897, 299, 598, 26, 962, 481, 37, 407, 814, 74, 518, 259, 777, 111, 555, 185, 370, 740, 148, 592, 296, 888, 444, 222, 666, 333, 999, 27, 621, 207, 414, 828, 138, 552, 276, 92, 644, 322, 966, 483, 161, 805, 35, 595, 119, 357, 714, 238, 476, 952, 56, 784, 392, 196, 588, 294, 98, 686, 343, 49, 637, 91, 455, 910, 182, 364, 728, 104, 520, 260, 780, 130, 390, 78, 624, 208, 416, 832, 64, 320, 960, 60, 660, 220, 110, 440, 880, 176, 352, 704, 88, 968, 484, 242, 726, 363, 33, 561, 187, 374, 748, 68, 884, 442, 221, 663, 39, 741, 247, 494, 988, 52, 676, 338, 169, 845, 65, 715, 143, 429, 858, 286, 572, 44, 308, 616, 154, 770, 385, 77, 847, 121, 605, 55, 935, 85, 425, 850, 170, 340, 680, 136, 544, 272, 816, 408, 204, 102, 918, 459, 153, 306, 612, 36, 936, 312, 156, 468, 234, 702, 351, 117, 819, 273, 546, 42, 672, 224, 896, 448, 112, 336, 48, 192, 384, 768, 128, 512, 256, 32, 736, 368, 184, 920, 460, 230, 690, 345, 115, 575, 23, 391, 782, 46, 874, 437, 19, 361, 722, 38, 646, 323, 969, 51, 867, 289, 578, 34, 986, 493, 29, 377, 754, 58, 638, 319, 957, 87, 783, 261, 522, 174, 348, 696, 232, 928, 464, 116, 580, 290, 870, 435, 145, 725, 25, 625, 125, 875, 175, 525, 75, 375, 750, 250, 500, 1000, 200, 600, 100, 700, 350, 70, 490, 245, 980, 140, 280, 560, 40, 400, 800, 160, 640, 80, 240, 720, 90, 360, 120, 480, 96, 288, 576, 72, 864, 144, 432, 216, 24, 648, 162, 324, 54, 270, 810, 405, 135, 675, 225, 450, 150, 300, 900, 180, 540, 108, 972, 486, 243, 729, 81, 567, 189, 945, 315, 63, 441, 882, 147, 735, 105, 630, 126, 378, 756, 252, 504, 168, 840, 210, 420, 84, 924, 462, 231, 693, 99, 891, 297, 594, 198, 396, 792, 264, 132, 528, 66, 330, 990, 495, 165, 825, 275, 550, 50, 650, 325, 975, 195, 585, 45, 765, 255, 510, 30, 570, 285, 855, 171, 513, 57, 627, 209, 418, 836, 76, 532, 266, 798, 399, 133, 665, 95, 475, 950, 190, 380, 760, 152, 608, 304, 912, 114, 456, 228, 684, 342, 18, 738, 246, 492, 984, 328, 656, 164, 820, 410, 205, 615, 123, 369, 41, 451, 902, 82, 574, 287, 861, 21, 609, 203, 406, 812, 28, 868, 434, 217, 651, 93, 837, 279, 558, 186, 372, 744, 248, 992, 496, 124, 620, 310, 930, 465, 155, 775, 31, 341, 682, 62, 806, 403, 13, 949, 73, 657, 219, 438, 876, 292, 584, 146, 730, 365, 5, 505, 101, 404, 808, 202, 606, 303, 909, 9, 963, 321, 107, 856, 428, 214, 642, 6, 426, 213, 639, 71, 355, 710, 142, 568, 284, 852, 12, 996, 332, 664, 166, 498, 249, 747, 83, 415, 830, 10, 970, 485, 97, 873, 291, 582, 194, 388, 776, 2, 524, 262, 786, 393, 131, 917, 7, 763, 109, 981, 327, 654, 218, 872, 436, 4, 508, 254, 762, 381, 127, 889, 1, 721, 103, 927, 309, 618, 206, 412, 824, 8, 904, 452, 226, 678, 339, 113, 791