On Partitions of the Positive Integers With No x, y, z Belonging to Distinct Classes Satisfying $x + y = z$

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1. Introduction

Let $\mathcal{P}^s = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_s$ be a partition of the positive integers into s non-empty classes. Let n be a positive integer and denote by $\mathcal{P}_n^s$ a partition, as above, of the integers 1, 2, ..., n.

The question whether for any given $\mathcal{P}_n^3$ the equation

$$x + y = z \tag{1}$$

has a solution with x, y, z belonging to the same class has been answered in the affirmative for sufficiently large n, by Schur [2].

To consider the opposite point of view, let us call a partition as above admissible, if the equation (1) has no solution with x, y, z belonging to three distinct classes. Then, imposing some conditions on $\mathcal{P}_n^3$ a solution, as above, of (1) can be forced.

For instance, C.J. Smyth [3] raised a question and V.E. Alekseev and S. Savchev [1] recently proposed it, as a problem in Kvant's contest, to prove:

Proposition 1. *There is no admissible* $\mathcal{P}_{3n}^3$ *with* $|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{A}_3|$.

E. and G. Szekeres [4] proved the same assertion from a weaker assumption, namely:

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[1] This research was completed while the author was a visiting professor at The University of Calgary.

Proposition 2. *There is no admissible* $\mathcal{P}_n^3$ *with* $\min(|\mathcal{A}_1|,|\mathcal{A}_2|,|\mathcal{A}_3|) > \frac{1}{4}n$.

In Section 3 of this paper, we shall study the admissible partitions $\mathcal{P}^3$. Our result, Theorem 2, implies that one of the classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ is a subset of the multiples of a certain integer greater than 1. As an application, Proposition 2 appears to be a corollary of that fact.

It turns out that the properties of an admissible partition $\mathcal{P}^3$ depend on a very particular structure, a nontrivial subset of the integers 1, 2, ..., n - 1 may have. This structure will be defined and investigated in Section 2.

Our main results concern the general case $\mathcal{P}^s$ and $\mathcal{P}_n^s$, and appear in Sections 4 and 5.

Theorem 2 admits a straightforward generalization while Proposition 2 becomes: There is no admissible $\mathcal{P}_n^s$ with $\min_i |\mathcal{A}_i| > 2^{1-s} n$.

As a so called Anti–Ramsey result, it is shown (Theorem 3) that if $\mathcal{P}_n^s$ is admissible, then $s \le \log_2 n + 1$ and equality can occur.

2. On $a^{(m)}$–Coverings of Subsets of $[1,m-1]$

Small letters shall denote positive integers, (x, y) is the g.c.d. of x and y. Capitals shall denote sets of positive integers. The set $\{1,2,\dots,n\}$ will be denoted $[1,n]$.

2.1 Definitions, notation and main properties

We shall be interested in properties of a subset S of $[1,m-1]$, when the following three structures are superposed.

(i) Central symmetry in $[1,m-1]$:

$$x \in S \Rightarrow m - x \in S \tag{2}$$

(ii) Periodicity modulo a:

$$a \in S \tag{3}$$

$$x \in S \Rightarrow x + a \in S \text{, provided } x + a < m \tag{4}$$

$$x - a \in S \text{, provided } x - a > 0$$

(iii) Central symmetry in $[1,a-1]$:

$$x \in S \Rightarrow a - x \in S \text{ , provided } a - x > 0 \tag{5}$$

Definition 1. *A subset S of* $[1, m-1]$ *is called* $a^{(m)}$*–covered if* (2), (3), (4), *and* (5) *hold.*

Example 1. Let x, y, a, m be integers $0 < a < m$. Define:

$$T = \{xa + ym;\, 0 < xa + ym < m\}, \tag{6}$$

then T is $a^{(m)}$–covered.

An important role in obtaining the results of Section 3 is played by the following Theorem 1 and its corollaries.

Theorem 1. *Let S be a subset of* $[1, m-1]$, $a \in S$ *and* $(a, m) = d$. *Then if S is* $a^{(m)}$*–covered it is also* $d^{(m)}$*–covered.*

Corollary 1. *If* a *is the minimal element for which* S *is* $a^{(m)}$*–covered and if* S *is also* $c^{(m)}$*–covered then* c *is a multiple of* a.

Proof of Corollary 1. Suppose $(a, b) = c < a$. Set $T = S \cap [1, b-1]$ then T is $a^{(b)}$–covered, therefore also $c^{(b)}$–covered, it follows by periodicity modulo b that S is $c^{(m)}$–covered. A contradiction.

Corollary 2. *If* $S \subset [1, m-1]$ *is* $a^{(m)}$*–covered then in* $\bar{S}$, *the complement of* S *in* $[1, m-1]$, *there is no* b *for which* $\bar{S}$ *is* $b^{(m)}$*–covered.*

Proof of Corollary 2. Suppose w.l.o.g. $a > b$, then $a - b$ must be in $\bar{S}$ otherwise $b \in S$. But $(a - b) \in \bar{S}$ implies $a \in \bar{S}$ a contradiction.

2.2 Proof of Theorem 1

One can suppose that m is not a multiple of a. Consider $m = at + r$, with $0 \le r < a$, then for some s, u, v $a = sd$, $m = ud$ and $r = ud$, moreover $(v, s) = 1$.

A first step in the proof is to show that S contains all positive multiples of d which are smaller than m. The next and final step is to show that if x is positive and smaller than d and for some l the integer $ld + x$ is a member of S, then $id \pm x$ is also a member of S for every i such that $id \pm x$ is positive and smaller than m. In fact, because of the periodicity modulo a, it is enough to prove the above statements replacing m by a.

Define now the integers a_k, b_k, c_k by:

$$a_k = \left(s\left\lceil \frac{kv}{s} \right\rceil - kv\right)d \quad \text{for } k = 1,2,\dots,s-1 \quad a_0 = 0,\ a_s = a \tag{7}$$

$$b_k = \left(kv - \left\lfloor \frac{kv}{s} \right\rfloor s\right)d \quad \text{for } k = 1,2,\dots,s-1 \quad b_0 = a,\ b_s = 0 \tag{8}$$

$$c_k = a_k + \left(t + \left\lfloor \frac{v(k+1)}{s} \right\rfloor - \left\lceil \frac{kv}{s} \right\rceil\right)a \quad \text{for } k = 0,1,\dots,s-2. \tag{9}$$

A direct verification shows that

$$a_k + b_k = a, \quad m - a < c_k < m \tag{10}$$

$$b_{k+1} + c_k = m \tag{11}$$

$$b_k = a_{s-k} \tag{12}$$

and for $k = 1,2,\dots,s-1$, $0 < a_k < a$, $0 < b_k < a$, moreover, using $(v, s) = 1$ and (12) it is easy to see that

$$\{a_k\}_{k=1}^{s} = \{b_k\}_{k=0}^{s-1} = \{id\}_{i=1}^{s}. \tag{13}$$

The first step in the proof of Theorem 1 is done by proving the following lemma and observing that a_s and b_0 are members of S.

Lemma 1. *The sets* $\{a_k\}_{k=1}^{s-1}$, $\{b_k\}_{k=1}^{s-1}$ *and* $\{c_k\}_{k=0}^{s-2}$ *are subsets of* S.

Proof of Lemma 1. The integer c_0 is an element of S, since it is a multiple of a, consequently by (11) and (2) b_1 is an element of S and therefore, by (10) and (5) also $a_1 \in S$. Apply induction and suppose $c_j \in S$, then as above b_{j+1} and a_{j+1} are elements of S, thus also c_{j+1} is an element of S, by (9) and (4).

Notice that the above first step in the proof of Theorem 1 shows that if a and m are given, then all positive multiples of d can be obtained from a by central symmetry in

$[1, m-1]$ and $[1, a-1]$ and periodicity modulo a. This is the converse of the statement in Example 1.

For the second step, let x be positive and smaller than d. Define $a_k(x) = a_k + x$, $b_k(x) = b_k + x$ and $c_k(x) = c_k + x$. With that notation, relation (13) is improved to

$$\{a_k(x)\}_{k=0}^{s-1} = \{id + x\}_{i=1}^{s-1}$$
$$\{b_k(x)\}_{k=1}^{s} = \{id + x\}_{i=1}^{s}.$$

The proof of the theorem is then achieved by proving the following lemma by arguments similar to those used in the proof of Lemma 1.

Lemma 2. *If for some l and x as above the integer $ld + x$ or $ld - x$ is an element of S then $\{a_k(x)\}_{k=0}^{s-1}$ and $\{b_k(x)\}_{k=1}^{s}$ are subsets of S.*

Definition 2. A subset S of $[1, m-1]$ is called *bad* if S is $a^{(m)}$–covered for some a. The multiples of a, which are members of S are called bad elements. The subsets of S containing all members of S which are not bad will be denoted S'.

We shall use the above defined concepts in the next section and in Section 5.

3. On Admissible Tripartitions of the Positive Integers

Let $\mathcal{P}^3 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ be an admissible partition of the positive integers. Then each of the classes contains a smallest element; let us denote the largest of them by m. We shall choose the notation so that $\mathcal{A} \supset 1$ and $\mathcal{C} \supset m$. Notice that $m \geq 4$.

Define $A = \{a;\ \ a \in \mathcal{A} \text{ and } a < m\}$ and similarly $B = \{b;\ \ b \in \mathcal{B} \text{ and } b < m\}$.

Proposition 3. (i) *A and B are complementary in* $[1, m-1]$.

(ii) *A and B both have the central symmetry property in* $[1, m-1]$.

(iii) *A and B can not be both bad.*

Proof. For (i), $A \cup B$ contains all positive integers, smaller than m. For (ii), the contrary contradicts the admissibility of $\mathcal{P}$. For (iii), the statement is implied by Corollary 2 and (i).

Further structural properties of A and B are established in the following proposition.

Proposition 4. *If c is the minimal bad element, then*

(i) $c \geq 2$

(ii) *for $c = 2$ it follows that $m = 2w$ for some w, B is bad and* $B = \{2i\}_{i=1}^{w-1}$

(iii) *for $c = 3$ it follows that $m = 3v$ for some v, B is bad and* $B = \{3i\}_{i=1}^{v-1}$.

Proof. For (i), otherwise $B = \phi$.

For (ii), A contains 1; if $2 \in A$ and 2 is bad then $A = [1, m-1]$ and $B = \phi$ which is impossible.

For (iii), A contains 1; if $3 \in A$ and 3 is bad then $2 \in A$ and again $B = \phi$.

The main result of this section, the following theorem, can be now formulated.

The long assumption in the theorem is only to be self–contained. It can be replaced by: "using the above defined notation."

Theorem 2. *Let $\mathcal{P}^3 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ be an admissible partition of the positive integers and let m, the smallest element of $\mathcal{C}$, be larger than the smallest element of $\mathcal{A}$ and of $\mathcal{B}$. Let, moreover, A and B be the subsets of $\mathcal{A}$ and $\mathcal{B}$ respectively containing the elements smaller than m. Furthermore, denote by A', respectively B', the subsets of A, respectively B, containing all the elements which are not bad, then*

$$\mathcal{A} \supset A \cup \{A' + im\}_{i=1}^{\infty} \tag{13a}$$

and

$$\mathcal{B} \supset B \cup \{B' + im\}_{i=1}^{\infty}. \tag{13b}$$

Proof.

Case 1. $A = A'$ i.e., A is not bad.

a) First we shall prove that

$$a \in A \Rightarrow a + m \in \mathcal{A}$$

Since A is not bad a is not a bad element, hence one of the following two must hold

(i) $a = a' + b$, $a' \in A$, $b \in B$

(ii) $\exists i$, $\exists a'$ such that $a' + ja \in A$ for $j = 0, 1, \ldots, i-1$, but $a' + ia \in B$.

If (i) holds then $a' = a - b$; and we claim that $a' + m \in \mathcal{A}$. Indeed,

$$a' + m = a + (m - b),$$

the left side is not in $\mathcal{B}$, by the admissibility of $\mathcal{P}$, while the right side is not in $\mathcal{C}$, by the same reason, noticing that $m - b$ is in B by central symmetry. Since $\mathcal{P}$ is a partition $a' + m$ must be in one of the classes so it is in $\mathcal{A}$.

Now consider $a + m$:

$$a + m = (a' + m) + b.$$

As above, the left side can not be in $\mathcal{B}$, while the right side can not be in $\mathcal{C}$, since $a' + m$ is in $\mathcal{A}$, so $a + m \in \mathcal{A}$

If (ii) holds then $a + m$ can be written as

$$a + m = [m - a' - (i-1)a] + [a' + ia),$$

the left side shows that $a + m$ is not in $\mathcal{B}$; while the right side shows that it is not in $\mathcal{C}$, since the first brackets are in $\mathcal{A}$ and the second in $\mathcal{B}$. So $a + m$ is in $\mathcal{A}$.

(b) Now we shall prove

$$a \in A \Rightarrow a + jm \in \mathcal{A} \text{ for } j = 1, 2, \ldots \tag{14}$$

The proof is by induction. For $j = 1$ (14) is true, suppose true for $j - 1$. It follows that $jm - a \in \mathcal{A}$ since $jm - a = (m - a) + (j-1)m$. Now, if a is as in (i), then

$$a' + (j-1)m + m = a' + jm = [a + (j-1)m] + (m - b)$$

therefore, $a' + jm$ is in $\mathcal{A}$; then by

$$a + (j-1)m + m = a + jm = (a' + jm) + b$$

it follows that $a + jm$ is also in $\mathcal{A}$.

If a is as in (ii), then

$$a + (j-1)m + m = a + jm = [jm - (a' + (i-1)a)] + (a' + ai),$$

hence $a + jm$ is in $\mathcal{A}$.

Case 2. $A \neq A'$, i.e., A is bad.

The arguments used in this case are similar to those used in Case 1, but one has to avoid bad elements in A, on the other hand the arguments are even simpler since one can use the fact that B is not bad, and Case 1 applies to B.

a) Again we first prove

$$a \in A' \Rightarrow a + m \in \mathcal{A}. \tag{15}$$

Now a is not bad by assumption and (i) or (ii) holds.

If (i) holds, $a = a' + b$ as before and $a + m = a' + (b + m)$ showing that (15) holds.

For (ii), the proof is exactly as in Case 1.

b) We shall now prove the analogue of (14):

$$a \in A' \Rightarrow a + jm \in \mathcal{A}. \quad j = 1, 2, \ldots$$

This is true for $j = 1$, suppose by induction true for $j - 1$, then for (i),

$$[a + (j-1)m] + m = a + jm = a' + (b + jm)$$

where the left–hand bracket is in $\mathcal{A}$ by induction while the right–hand bracket is in $\mathcal{B}$ by Case 1. Therefore, $a + jm \in \mathcal{A}$

For (ii),

$$[a + (j-1)m] + m = a + jm = [m - (a' + (i-1)a)] + [(a' + ia) + (j-1)m]$$

proving that $a + jm \in \mathcal{A}$ Notice the brackets are not set in the same way as in Case 1.

This proves (13a). The proof of (13b) is similar.

We shall formulate below three corollaries exhibiting the effect on C, the obtained results have.

Corollary 3. *If* c *is the minimal bad element, then* $C \subset \{ic\}_{i=1}^{\infty}$.

Corollary 4. *If neither of* A, B *is bad then* $C \subset \{im\}_{i=1}^{\infty}$.

Corollary 5. *If in the admissible partition* $\mathcal{P}_n^3$ *neither one of* A *and* B *is bad, then*

$$|C| \leq \frac{1}{4} \cdot n \tag{16}$$

Proof. m is at least 4.

Proposition 2 is a consequence of the above results.

If none of A, B is bad, the proposition holds by Corollary 5. If one of A, B is bad but $c \geq 4$, then it follows from Corollary 3.

The remaining cases are $c = 2$ or 3. For $c = 2$, by Proposition 4, $m = 2w$, $B = \{2i\}_{i=1}^{w-1}$. Therefore, $|\mathcal{A}| \geq \frac{n}{2}$ and $\mathcal{B}$ and C cannot both satisfy the requirement. For $c = 3$, by Proposition 4, $m = 3v$, $\mathcal{B} = \{3i\}_{i=1}^{v-1} |\mathcal{A}| \geq \frac{2}{3} n$ and again $\mathcal{B}$ and C cannot both satisfy the requirement.

4. On Admissible Partitions $\mathcal{P}^s$ and $\mathcal{P}_n^s$

4.1 An anti-Ramsey theorem

If n is given, admissible partitions $\mathcal{P}_n^s$ cannot exist for too large s, for instance, if $s = n$ each class contains a single integer and if $x + y = z$ then x, y, z are from different classes. Hence it is a natural question to ask for the smallest s which enforces a solution in distinct classes. The answer will be formulated in Theorem 3.

Let $\mathcal{P}_n^s = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_s$ be an admissible partition of $[1, n]$. Denote the smallest element of $\mathcal{A}_i$ by m_i and choose the notation so that $m_1 < m_2 < \ldots < m_s$.

Then $m_1 = 1$ and for convenience put $m_s = m$.

Proposition 5. $m_{i+1} \geq 2m_i$, $i = 1, 2, \ldots, s-1$

Proof. Suppose, to the contrary, that $m_{i+1} < 2m_i$, then

$$0 < m_{i+1} - m_i < m_i ,$$
$$m_{i+1} - m_i \in \mathcal{A}_j , \quad j < i$$

and
$$(m_{i+1} - m_i) + m_i = m_{i+1}$$

contradicts the admissibility of $\mathcal{P}_n^s$.

Corollary 6. $$m \geq 2^{s-1}. \tag{17}$$

Theorem 3. (i) *If* $\mathcal{P}_n^s$ *is admissible, then*

$$s \leq 1 + \log_2 n$$

(ii) *there is an admissible* $\mathcal{P}_n^s$ *with* $s = 1 + \lfloor \log_2 n \rfloor$.

Proof. For (i), from (17) $s - 1 \leq \log_2 m \leq \log_2 n$.

For (ii), choose $m_i = 2^i$, $s = 1 + \lfloor \log_2 n \rfloor$ and

$$\mathcal{A}_i = \{ a \mid a \in [1, n], a \equiv m_i \ (\mathrm{mod}\ m_{i+1}), \ i \in [1, s-1] \}$$
$$\mathcal{A}_s = \{ a \mid a \in [1, n], a \equiv 0 (\mathrm{mod}\ m_s) \}.$$

4.2 A generalization

We shall prove the following generalization of Proposition 2.

Theorem 4. *No admissible* $\mathcal{P}_n^s$ *exists with* $\min_i | \mathcal{A}_i | > 2^{1-s} n$.

We are adopting a definition and using a notation as in [4]:

Let the set $\mathcal{A}_s$ be $\mathcal{A}_s \{z_1, z_2, \ldots, z_\zeta\}$ arranged in increasing order so $z_1 = m_s = m$, and define r to be the smallest difference between (successive) members of $\mathcal{A}_s$ and let k be the smallest suffix for which $z_{k+1} - z_k = r$.

The following lemma is an immediate generalization of a lemma used in [4]:

Lemma 3. $r \geq m_{s-1}$.

Proof. The proof is also a straightforward generalization of that in [4]; we shall give the proof only for completeness, [4] being not too accessible.

Suppose $r < m_{s-1}$. Then $r \in \mathcal{A}_j$ with $j < s-1$ and $m_{s-1} - r \in \mathcal{A}_l$, also with $l < s-1$. Observe that by admissibility

$$t < s-1,\ \mathcal{A}_t \not\ni z_k + r - m_{s-1} = z_k - (m_{s-1} - r) \notin \mathcal{A}_{s-1}$$

therefore $z_k + r - m_{s-1} \in \mathcal{A}_s$ and consequently

$$t < s-1,\ \mathcal{A}_t \not\ni z_k - m_{s-1} = (z_k + r - m_{s-1}) \notin \mathcal{A}_{s-1}$$

therefore $z_k - m_{s-1} \in \mathcal{A}_s$

The difference between these above defined members of $\mathcal{A}_s$ being r and both being smaller than z_k leads to a contradiction.

Observation. Defining analogously the smallest difference r_i in set $\mathcal{A}_i$ one can prove in the same way that $r_i \geq m_{i-1}$.

Proof of Theorem 4.
Case 1. $r \geq s^{s-1}$.

In this case

$$n \geq z_\zeta \geq m + (\zeta - 1)2^{s-1} \geq \zeta 2^{s-1}$$

therefore $\min_i \mathcal{A}_i \leq \zeta \leq 2^{1-s} n$.

Notice that in the remaining cases $m_{s-1} \leq r < 2^{s-1}$, and since, as shown below, Case 2 cannot happen, one has $r = m_{s-1}$.

Case 2. $2^{s-1} > r = m_{s-1} + \beta,\ 0 < \beta < m_{s-1}$.

Let us denote $\bigcup_{i=1}^{s-z} \mathcal{A}_i$ by U. Then the integers $[z_k + 1, z_k + r - 1]$ are not in $\mathcal{A}_s$, since $z_{k+1} = z_k + r$.

Moreover, the integers $[z_k + 1, z_k + m_{s-1} - 1]$ are also not in $\mathcal{A}_{s-1}$, since the integers $[1, m_{s-1} - 1]$ being smaller than m_{s-1} are in U. In particular, β is in U and $z_k + \beta$

is not in $\mathcal{A}_s$ and not in $\mathcal{A}_{s-1}$. So it must be in U but this is impossible since $(z_k + \beta) + m_{s-1} = z_{k+1}$ contradicts the admissibility.

<u>Case 3</u>. $r = m_{s-1}$.

In this case we shall prove first that all non-multiples of r are in U.

Observe that $[z_k+1,\ z_k+r-1] \cap \mathcal{A}_s = \phi$ since $z_{k+1} = z_k + r$. Furthermore,

$$[1, r-1] \subset U \tag{18}$$

since the first integer not in U is $m_{s-1} = r$. Therefore $[z_k+1,\ z_k+r-1] \cap \mathcal{A}_{s-1} = \phi$ and $[z_k+1,\ z_k+r-1] \subset U$. Moreover $[z_k-1,\ z_k-m+1] \subset U$ since none of the elements is in $\mathcal{A}_s$ and none is in $\mathcal{A}_{s-1}$.

We shall prove our claim using induction.

By (18) one can start induction. Let $0 < \mu < r$ and suppose $qr + \mu \in U$ for $q = 1, 2, \ldots, t-1$. Consider $tr + \mu$. If

$$z_k - [(t-1)r + \mu] > 0 \tag{19}$$

then
$$z_k - [(t-1)r + \mu] = (z_k - r) - [(t-2)r + \mu]. \tag{20}$$

The left-hand side shows that this integer cannot be in $\mathcal{A}_{s-1}$ while the right-hand side shows that it cannot be in $\mathcal{A}_s$ either, since the first term must be in $\mathcal{A}_{s-1}$. So it is in U. But this implies that

$$t_{r+\mu} = (z_k + r) - (z_k - (t-1)r - \mu) = [(t-1)r + \mu + r]$$

must be in U. Indeed middle terms show that it cannot be in $\mathcal{A}_{s-1}$ and the right-hand side shows that it cannot be in $\mathcal{A}_s$.

If the expression in (19) is negative, then

$$(t-1)r + \mu - z_k$$

turns out to be U by simply reversing the order of the terms in the differences in (20) and finally

$$tr + \mu = (z_k + r) + [(t-1)r + \mu - z_k] = [(t-1)r + \mu] + r$$

shows that all non-multiples of r are in U.

This proves the theorem also in Case 3, hence it follows that $|U| \geq \frac{r-1}{r} n$ but then

$$|\mathcal{A}_{s-1} \cup \mathcal{A}_s| \leq \frac{1}{r} n = \frac{1}{m_{s-1}} n \leq 2^{z-s} n$$

and therefore not both $|\mathcal{A}_{s-1}|$ and $|\mathcal{A}_s|$ can be larger than $2^{1-s} n$.

5. Some Structures in $\mathcal{P}^s$

The structural properties of $\mathcal{P}^3$ based on concepts introduced in Section 2 and given in Section 3 generalized straightforward for $\mathcal{P}^s$ for general s. Here the formulation of the necessary definitions and results.

Define $A_i = \mathcal{A}_i \cap [1, m-1]$ for $i \in [1, s-1]$ while A_i' is the set of non–bad elements of $\mathcal{A}_i$.

Proposition 3 becomes:

Proposition 6. *If $\mathcal{P}^s$ is an admissible partition of the integers, then*

(i) $\bigcup_{i=1}^{s-1} A_i = [1, m-1]$

(ii) A_i *has the central symmetry property in* $[1, m-1]$ *for each* $i \in [1, s-1]$

(iii) A_i *can be bad for at most a single value of* $i \in [1, s-1]$

The analogue to Theorem 2 is:

Theorem 5. *If $\mathcal{P}^s$ is an admissible partition of the integers then for each i, $i \in [1, s-1]$*

$$\mathcal{A}_i \supset A_i + \{A_i' + jm\}_{j=1}^{\infty}.$$

The proofs are omitted.

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