The following Lemma can be derived in many ways. Euler proved it (see [2]) and it also a special case of Equation 5.42 in [3]. We present a combinatorial proof due to [1].

(The lemma AFTER that is the one we wonder if it is known.)

Lemma 0.1 *Let $p \in \mathbb{Z}[x]$ be a polynomial of degree $\le n-1$. Let $s \in \mathbb{N}$, $s \ge 1$. Then*

$$\sum_{i=0}^{n} p(s+i)\binom{n}{i}(-1)^i = 0$$

Proof:

We first prove that, for any $m, n, s \in \mathbb{N}$ with $m < n$,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(s+i)^m = 0.$$

Consider the following problem:

How many ordered m-tuples of elements of $\{1, \ldots, n+s\}$ are there such that each element of $\{1, \ldots, n\}$ appears at least once?

This problem is as easy as it looks. The answer is 0.

However, we can also solve this problem a different way. We solve it by inclusion-exclusion.

How many ordered tuples are there with no constraints: $(s+n)^m$.

We subtract out those that do not use 1 or do not use 2 or $\cdots$ or do not use n? There are $\binom{n}{1}(s+n-1)^m$ of these.

We then add back those that used two of $\{1, \ldots, n\}$. There are $\binom{n}{2}(s+n-2)^m$ of these.

We keep doing this to obtain

$$0 = (s+n)^n + (-1)^1\binom{n}{1}(s+n-1)^m + (-1)^2\binom{n}{2}(s+n-2)^m + \cdots + (-1)^n\binom{n}{n}(s+n-n)^m$$

If n is even this gives the result we seek. If n is odd then negate both sides and we obtain the result we seek.

We now proof the Lemma.

Let

$$p(x) = \sum_{j=0}^{n-1} a_j x_j.$$

Then

$$\begin{aligned}
\sum_{i=0}^{n} p(s+i)\binom{n}{i}(-1)^i &= \sum_{i=0}^{n}\sum_{j=0}^{n-1} a_j (s+i)^j \binom{n}{i}(-1)^i \\
&= \sum_{j=0}^{n-1} a_j \sum_{i=0}^{n} (s+i)^j \binom{n}{i}(-1)^i
\end{aligned}$$

By the above all of the inner sums are 0. Hence the entire sum is 0. ∎

Lemma 0.2 *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial of degree n with constant term 0. Then*

$$p(s) - \sum_{k=1}^{n} \binom{s+k-1}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \left(p(s+i+1) - p(s+i)\right) \;=\; 0\,.$$

Proof:

$$\begin{aligned}
& p(s) - \sum_{k=1}^{n} \binom{s+k-1}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \left(p(s+i+1) - p(s+i)\right) \\
= \quad & p(s) + \sum_{k=1}^{n} \binom{s+k-1}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i p(s+i) \\
= \quad & \sum_{k=0}^{n} \binom{s+k-1}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i p(s+i) \\
= \quad & \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^i p(s+i) \binom{s-1+j}{j} \binom{j}{i} \qquad \text{collecting the } p(s+i) \text{ terms together, for fixed } i \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \sum_{j=i}^{n} \binom{s-1+j}{j} \binom{j}{i} \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \sum_{j=i}^{n} \binom{s-1+j}{s-1+i} \binom{s-1+i}{i} \qquad \text{by a version of trinomial revision} \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \binom{s-1+i}{i} \sum_{j=i}^{n} \binom{s-1+j}{s-1+i} \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \binom{s-1+i}{i} \sum_{j=0}^{n-i} \binom{s-1+i+j}{s-1+i} \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \binom{s-1+i}{i} \sum_{j=0}^{n-i} \binom{s-1+i+j}{j} \\
= \quad & \sum_{i=0}^{n} (-1)^i p(s+i) \binom{s-1+i}{i} \binom{s-1+i+(n-i+1)}{n-i} \qquad \text{by parallel summation}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n}(-1)^i p(s+i)\binom{s-1+i}{s-1}\binom{s+n}{n-i} \\
&= \sum_{i=0}^{n}(-1)^i p(s+i)\frac{s}{s+i}\binom{s+i}{s}\binom{s+n}{n-i} && \text{by extraction} \\
&= \sum_{i=0}^{n}(-1)^i p(s+i)\frac{s}{s+i}\binom{s+i}{s}\binom{s+n}{s+i} \\
&= \sum_{i=0}^{n}(-1)^i p(s+i)\frac{s}{s+i}\binom{n}{i}\binom{s+n}{n} && \text{by trinomial revision} \\
&= \binom{s+n}{n} s \sum_{i=0}^{n}\frac{p(s+i)}{s+i}\binom{n}{i}(-1)^i \\
&= 0 \qquad \text{by Lemma 0.1}
\end{aligned}
$$

The last equality holds by noting that $\frac{p(s+i)}{s+i}$ is a polynomial of degree $n-1$ and applying Lemma 0.1. ▮

We thank Doron Zeilberger for pointing out reference [2] to us. We also thank the author of [1] whoever that may be.

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