# Decidability of WS1S and S1S: An Exposition

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# Credit Where Credit is Due

Buchi proved that WS1S was decidable.
I don't know off hand who proved S1S decidable.

# Part I
## We Define WS1S And Prove It's Decidable

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**WS2S** Weak second order with **two** Successors- two ways to add to a string. Basic objects are strings of 0's and 1's.

**WS2S** is also decidable but we will not prove this.

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5. For any $c \in \mathbb{N}$, $X = Y + c$ is an Atomic Formula.
   This means that $X = \{y + c : y \in Y\}$.

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$$(Q_1 v_1)(Q_2 v_2)\cdots(Q_m v_m)[\phi(v_1,\ldots,v_n)]$$

where the $Q_i$'s are quantifiers, the $v_i$'s are either numbers or finite-set variables, and $\phi$ has no quantifiers. ($m$ quantifiers, $n \geq m$ vars. This is a formula– could be vars that are not quantified over.)

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3. $(Q_1 x)[\phi_1(x)] \vee (Q_2 y)[\phi_2(y)]$ is equiv to $\neg((\neg Q_1 x)[\phi_1(x)] \wedge (\neg Q_2 y)[\phi_2(y)])$.

# Key Definition

**Def** If $\phi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ is a WS1S Formula then $\mathrm{TRUE}(\phi)$ is the set

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This is the set of $(a_1, \ldots, a_n, A_1, \ldots, A_m)$ that make $\phi$ $\mathrm{TRUE}$.

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The triple $(3, 4, \{0, 1, 2, 4, 7\})$ is represented by

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | 1 | $*$ | $*$ | $*$ | $*$ |
| $y$ | 0 | 0 | 0 | 0 | 1 | $*$ | $*$ | $*$ |
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**Note** After we see 0001 for $x$ we **do not care** what happens next. The *'s can be filled in with 0's or 1's and the string of symbols from $\{0,1\}^3$ above would still represent $(3, 4, \{0, 1, 2, 4, 7\})$.

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Finite set $X$ is represented by a string in $\{0,1\}^*$ which is its bit-vector.

# Example And Our Alphabet

Consider the set

$$\{(x, y, X) : (x = y + 1) \wedge (y \in X)\}$$

We want to show that it's regular. Here is an example of how we **represent** a tuple (number,number,finite set):

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| $X$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |

This string is IN our lang since $x = 5$, $y = 4$, and $X = \{0, 1, 2, 4, 7\}$.

Alphabet is $\{000, 001, 010, 011, 100, 101, 110, 111\}$ though we think of it vertically rather than horizontally.

## Stupid Strings

What does

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This string is **Stupid!** There is no value for $x$. This string does not represent anything!

Our DFA's will have 3 kinds of states: **accept, reject,** and **stupid. Stupid** means that the string did not represent anything because it has a number-variable be all 0's. (It is fine for a set var to of all 0's- that would be the empty set.)

# Key Theorem

**Thm** For all WS1S formulas $\phi$ the set $\mathrm{TRUE}(\phi)$ is regular.

We prove this by induction on the formation of a formula. If you prefer- induction on the length of a formula.

# Theorem for Atomic Formulas

**Lemma** For all WS1S atomic formulas $\phi$ the set $\mathrm{TRUE}(\phi)$ is regular.

On the next few slides we give the DFA for some Atomic Formulas. The ones we do not may be HW or on the Final.

# For any $c \in \mathbb{N}$, $x = y + c$ is an Atomic Formula

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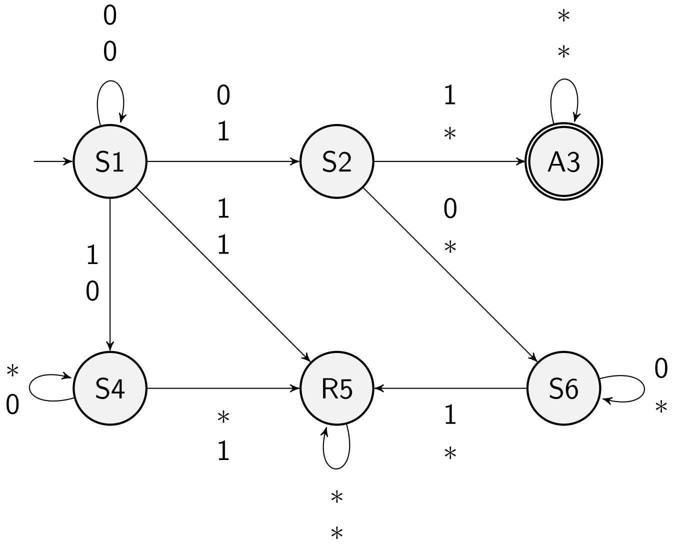
The DFA for $x = y + c$ is similar. Might be on a HW or the Final.

## $x = y + 1$



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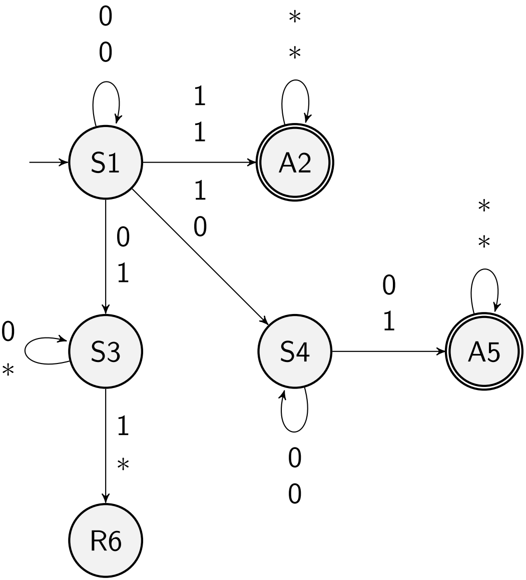

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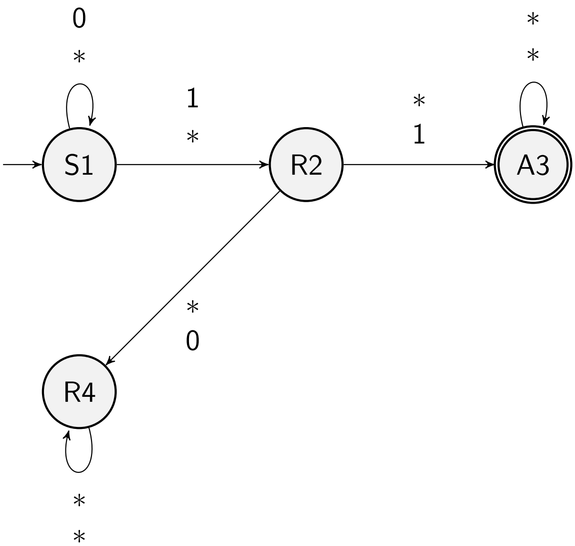

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Getting DFA's for those atomic formulas, or special cases, might be on a HW or the Final.

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Next slides for what to do about quantifiers.

# Theorem for Formulas (II)

$\mathrm{TRUE}(\phi(x_1, \ldots, x_n, X_1, \ldots, X_m))$ is regular.
We want $\mathrm{TRUE}((\exists x_1)[\phi(x_1, \ldots, x_n, X_1, \ldots, X_m)])$ is regular.
Ideas?

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Ideas?
Use nondeterminism.
Will show you in class.

# DFA Decidability Theorem

We need the following easy theorem:

**Thm** The following problem is decidable: given a DFA determine if **there exists** a string it accepts.

# DFA Decidability Theorem Proof

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**Might be on HW.**

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6. If $L(M) \neq \emptyset$ then $(\exists X)[\phi(X)]$ is TRUE.
   If $L(M) = \emptyset$ then $(\exists X)[\phi(X)]$ is FALSE.

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We will do the following **together.**

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**Note** No De Morgans Law–we complement the DFA.

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If YES then the original sentence is TRUE.
If NO then the original sentence is FALSE.

# Complexity of the Decision Procedure

Given a sentence

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And the answer is: Can do better: $\mathbf{2^{2^{n^3 \log n}}}$. **This is provably the best you can do (roughly).**

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Are there interesting problems that can be STATED in WS1S?
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**YES** Extensions of WS1S are used in low-level verification of code fragments. The MONA group has coded this up and used it, though their code uses MANY tricks to speed up the program in MOST cases.

**NO** There are no interesting MATH problems that can be expressed in WS1S.

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Presb Arith is decidable by TRANSFORMING Pres Arith Sentences into WS1S sentences.
Presb Arithmetic has been used in Code Optimization (using a better dec procedure than reducing to WS1S).

PART II OF THIS TALK:
WE DEFINE S1S AND PROVE IT'S DECIDABLE

# What is S1S?

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**Question** Can we still use finite automata?

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**Def** A ***B*-NFA** is an NFA. If $x \in \Sigma^\omega$ then $x$ is accepted by $B$-NFA $M$ if there is a path such that $M(x)$ hits a final state inf often.
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1. $B$-reg closed: UNION, INTER, PROJ.
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**Need** $B$-reg closed under complementation.

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**Odd News** Proof Uses **Ramsey Theory,** yet I never proved it in my Ramsey Theory course.

# $B$-Reg and $Mu$-Reg

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**Easy (IN GROUPS)** *Mu*-reg Closed: UNION, INTER, COMP.

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- $B$-reg easily closed: $\cup$, $\cap$, PROJ, but COMPLEMENT hard.
- $Mu$-reg easily closed: $\cup$, $\cap$, COMPLEMENT. But PROJ hard.
- How to prove? Show $B$-reg $=$ $Mu$-reg.

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$$(Q_1 X_1) \cdots (Q_n X_n)(Q_{n+1} x_1) \cdots (Q_{n+m} x_m)[\phi(x_1, \ldots, x_m, X_1, \ldots, X_n)]$$

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4. Construct B-NFA $M$ for $\{X : \phi(X) \text{ is true}\}$.
5. Test if $L(M) = \emptyset$.
6. If $L(M) \neq \emptyset$ then $(\exists X)[\phi(X)]$ is TRUE.
   If $L(M) = \emptyset$ then $(\exists X)[\phi(X)]$ is FALSE.

# COMPLEXITY OF THE DECISION PROCEDURE

Given a sentence

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How long will the procedure above take in the worst case?
$2^{2^{\cdots n}}$ steps since we do $n$ nondet to det transformations.

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Are there interesting problems that can be STATED in S1S?
**YES** Verification of programs that are supposed to run forever like operating systems. Verification of security protocols.

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**YES** Verification of programs that are supposed to run forever like operating systems. Verification of security protocols.

**NO** There are no interesting MATH problems that can be expressed in S1S.

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WS1S, S1S are about strings $0^*1$ and sets of such strings.

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# $\omega$-Reg

**Def** A language $L$ is **$\omega$-reg** if there exists regular langs $U_1, U_2, \ldots, U_n,\ V_1, V_2, \ldots, V_n$ such that

$$L = \bigcup_{i=1}^{n} U_i V_i^{\omega}.$$

**Thm** $B$-reg $= \omega$-reg

**Work with Neighbors**

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2. A language $L$ is **ioPrefix-reg** if there exists regular langs $U_1, U_2, \ldots, U_n,\ V_1, V_2, \ldots, V_n$ such that

$$L = \bigcup_{i=1}^{n} U_i \cdot \mathrm{ioPrefix}(V)$$

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William Gasarch-U of MD