Adjusted Winner
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1 Introduction

We discuss the *Adjusted Winner (AW)* method to divide a discrete set of items.

2 An Example and an Explanation

We give an example of the method. In the next section we give a formal algorithm that takes care of all cases.

Assume that Alice and Bob need to split the following items:
PICASSO, CAR, HOUSE, MILLION DOLLARS, ISLAND.

1) Each of them allocates 100 points between the five items indicating how much they like it. We assume they do not know each others tastes.

We assume they allocate as follows:

Item	Alice	Bob
PICASSO	15	40
CAR	20	15
HOUSE	20	15
MILLION	25	30
ISLAND	20	0

2) Each player gets the items they valued the most (for now). We ignore ties (for now). We indicate who gets what by a * and also add up the totals.

Item	Alice	Bob
PICASSO	15	40*
CAR	20*	15
HOUSE	20*	15
MILLION	25	30*
ISLAND	20*	0
TOTAL	60	70

So, for now,
Alice gets the Car, House, and Island
Bob gets the Picasso, and the Million.

Alice is called *The Loser* and Bob is called *The Winner*. This is only temporary.

Steps 1 and 2 are called *Phase 1*

3) Bob must give part of some item to Alice so that they can both have equal points. The item must be fluid (that is, it can be split). Lets consider the options

Bob gives Alice part of the Picasso (which we will consider fluid somehow). Note that Bob thinks the Picasso is worth 40 while Alice things the Picasso is worth 15. Hence its worth A LOT more to Bob then to Alice. We would likely end up having to give A LOT of it to Alice who didn't even want it that badly.

Bob gives Alice part of the Million. Note that Bob thinks the Million is worth 30 while Alice thinks the Million is worth 25. Hence its worth A LOT to Alice. We would likely not have to give that much to Alice.

So we give the Million. More formally we take the item a with the least ratio of (what Bob thinks its worth)/(What Alice thinks its worth).

Note that if Bob gave Alice x of his million its worth $25x$ to Alice and 30 to Bob. Hence we solve

$60 + 25x = 70 - 30x$

$55x = 10$

$x = \frac{10}{55} = \frac{2}{11}$.

SO Bob gives $\frac{2}{11}$ of the Million to Alice.

Alice ends up with Car, House, Island and 2/11 of the Million

Bob ends up with Picasso and 9/11 of the million.

They both end up with $60 + 25 \times (2/11) = 70 - 30 \times (2/11) \sim 64.54$.

Step 3 is called *Phase 2*.

3 General Procedure

The items are $I_1, \ldots, I_n$.

Item	Alice	Bob
I_1	a_1	b_1
I_2	a_2	b_2
$\vdots$	$\vdots$	$\vdots$
I_n	a_n	b_n

1) Initially give Alice all of the items I_i such that $a_i > b_i$ and Bob all the items I_i such that $a_i < b_i$. Put the items I_i such that $a_i = b_i$ aside for now.

2) Total the items. There are four scenarios

1. There were no items with $a_i = b_i$ and the totals are identical. Then we are done.

2. There are some items with $a_i = b_i$ and there is a way to divide them so that both parties have an equal number of points. We do that division and we are done. (Note: If the number of items is large then doing that division might be a difficult problem. For those who know the term, its an NP-complete problem.)

3. There are some items with $a_i = b_i$ and there is a no way to divide them so that both parties have an equal number of points. Divide the items so that the difference between Alice and Bobs total is as small as possible. (Note: If the number of items is large then doing that division might be a difficult problem. For those who know the term, its an NP-complete problem.) We now have allocated all of the items and the division is unequal. We proceed as if we are in the next case.

4. There were no items with $a_i = b_i$ and the totals are different. The person who has the larger sum we call *The Winner* and the person with the smaller sum we call *The Loser*. Find the fluid item that the Winner has that has the smallest ratio of (Value to Winner)/(Value to Loser). Let W be what the winner has, w be how the winner values the item, L be what the lower has, l be how the lower values the item.

   Solve the equation

$$W - wx = L + lx$$

   and take x of the item from the winner and give it to the loser.

   If (and this is usually the case) the totals are now equal then we are DONE. If not then $x = 1$ and you gave the entire item to the Loser and he's still a Loser!. Repeat the procedure with the next fluid item that the Winner has that has a the smallest ratio. Keep repeating until they have equal Totals. (If the number of fluid items is small this might not work.)

4 AW is Equitable

AW is equitable by design. At the end Alice thinks she has V and Bob thinks he has V.

5 AW is Proportional and Envy Free

Theorem 5.1 *AW is proportional.*

Proof: We use the notation of the general procedure.

Assume that at the end of AW Alice has items $I_1, \ldots, I_r$ and x of I_{r+1}, and Bob has $I_{r+2}, \ldots, I_n$ and $(1-x)$ of I_{r+1}. By the design of the protocol

$$a_1 + \cdots + a_r + xa_{r+1} = b_{r+2} + \cdots + b_n + (1-x)b_{r+1}$$

Assume, by way of contradiction, that either player has < 50. Since both players have the same, both have < 50. Hence

$a_1 + \cdots + a_r + xa_{r+1} < 50$

$b_{r+2} + \cdots + b_n + (1-x)b_{r+1} < 50$.

Add these two together to obtain

$a_1 + \cdots + a_r + a_{r+1} + b_{r+2} + \cdots + b_n < 100$

Since Alice originally got $a_1, \ldots, a_{r+1}$ we must have that, for all $1 \le i \le r+1$, $a_i \ge b_i$. Therefore

$a_1 + \cdots + a_r + a_{r+1} + b_{r+2} + \cdots + b_n \ge b_1 + \cdots + b_{r+1} + b_{r+2} + \cdots b_n = 100$.

Hence we have $100 < 100$ which is a contradiction. ∎

Since any 2-party protocol that is Proportional is also Envy-Free we have the following.

Corollary 5.2 *AW is Envy Free*

6 AW is Efficient

Let Alice and Bob divide a set of discrete goods using AW. We want to show that there is no other division that makes one of them better off and the other not worse off.

We need the following lemma which we do not prove.

Lemma 6.1 *Assume that an allocation of discrete goods is not efficient. Then there exists items I_1 (which Alice has part of) and I_2 (which Bob has part of) that are fluid and two fractions α, β such that if Alice gives α of I_1 to Bob, and Bob gives β of I_2 to Alice, then at least one of them is better off and the other one is not worse off.*

Theorem 6.2 *AW is efficient.*

Proof: Assume, by way of contradiction, that AW is not efficient. By Lemma 6.1 there exists I_1, I_2, α, β such that if Alice gives α of I_1 to Bob and Bob gives β of I_2 to Alice, then one is better off and the other is not worse off.

- a_1 is what Alice thinks I_1 is worth.
- a_2 is what Alice thinks I_2 is worth.
- b_1 is what Bob thinks I_1 is worth.
- b_2 is what Bob thinks I_2 is worth.

Hence we have the following table.

Item	Alice	Bob
I_1	a_1	b_1
I_2	a_2	b_2

Since Alice is no worse off giving away α of I_1 and getting β of I_2 we know that

$$\beta a_2 \geq \alpha a_1.$$

Since Bob is no worse off giving away β of I_2 and getting α of I_1 we know that

$$\alpha b_1 \geq \beta b_2.$$

We refer to the equations above as EQ1 and EQ2.

Since one of the two is better off, one of the EQ's has a strict inequality. We use this later.

We assume that Alice was the Winner in Phase I of AW. (The other case is similar.) Hence everything that she has she got in Phase I. Therefore $a_1 \geq b_1$.

There are two cases depending on if either I_2 was split.

Case 1: I_2 was given to Bob in Phase I. Hence $b_2 \geq a_2$.

By adding EQ1 and EQ2 and using that one of EQ1,EQ2 is strict we obtain

$$\beta a_2 + \alpha b_1 > \alpha a_1 + \beta b_2$$

which we call E3.

Multiply $a_1 \geq b_1$ by α and $b_2 \geq a_2$ by β, and add, to get

$$\alpha a_1 + \beta b_2 \geq \alpha b_1 + \beta a_2$$

$$\alpha a_1 + \beta b_2 \geq \beta a_2 + \alpha b_1$$

which we call E4.

Note that E3 and E4 contradict!

Case 2: Part of I_2 was given to Bob in Phase II. Hence $a_2 \geq b_2$ which is not at all helpful to the argument. But why was item I_2 used instead of item I_1? Because $\frac{a_2}{b_2} \leq \frac{a_1}{b_1}$. We rewrite this as

$$a_1 b_2 \geq a_2 b_1$$

which we call E5.

Multiply E1 by b_2 and E2 by a_2 to obtain:

$$\beta a_2 b_2 \geq \alpha a_1 b_2$$

$$\alpha a_2 b_1 \geq \beta a_2 b_2.$$

Hence

$$\alpha a_2 b_1 \geq \beta a_2 b_2 \geq \alpha a_1 b_2$$

Since one of these is a strict inequality we obtain

$$\alpha a_2 b_1 > \alpha a_1 b_2$$

$$a_2b_1 > a_1b_2.$$

This contradicts E5. ∎

7 AW is Super Cheat-Proof

What Alice thinks of her piece and Bob's piece is independent of Bob's honesty. So the protocol is super cheat proof.

Note that Bob could do better by lying, but Alice will still get at least half and will not be envious. The fool!

8 Extend to Three People? More?

In this section all calculations are only taken to 2 decimals.

Can AW be extended to three people? If we do that, which properties will still hold? We do an example as we describe the procedure.

1) Each of them allocates 100 points between the five items indicating how much they like it. We assume they do not know each others tastes.

We assume they allocate as follows:

Item	Alice	Bob	Carol
PICASSO	15	40	10
CAR	20	15	50
HOUSE	20	15	10
MILLION	25	30	0
ISLAND	20	0	30

2) Each player gets the items they valued the most (for now). We ignore ties (for now). We indicate who gets what by a * and also add up the totals.

Item	Alice	Bob	Carol
PICASSO	15	40*	10
CAR	20	15	50*
HOUSE	20*	15	10
MILLION	25	30*	0
ISLAND	20	0	30*
TOTAL	20	70	80

So, for now,
Alice gets the House. She has 20 points.
Bob gets the Picasso, and the Million. He has 70 points.
Carol gets the Car and the Island. She has 80 points.
Carol is called *The Winner* and Alice and Bob are called *The Losers*.

3) The two losers execute a procedure just like AW to ensure that they have equal values (though both will still be less than the Winner). They can either split the Picasso which has ratio $40/15 = 8/3$ or split the Million which has ratio $30/25 = 6/5$ which is smaller. So they split the Million

$$20 + 25x = 70 - 30x$$

$$55x = 50$$

$$x = 10/11$$

So Bob gives Alice 10/11 of the Million.
Alice gets the House and 10/11 of the Million. She has $20+25\times\frac{10}{11} = 42.72$.
Bob gets the Picasso, and the 1/11 of the Million. He has 42.72 (this is not an accident- we chose x so that they would be equal.
Carol gets the Car and the Island. She has 80 points.
Carol is still called *The Winner* and Alice and Bob are still called *The Losers*.

Now we take some item that Carol has and give some of it to Alice and Some of it to Bob. There is no real good way to pick this, so we'll pick the Car. Carol gives x of the Car to Alice and y of the Car to Bob. We will have to make sure that $x + y \leq 1$.
Alice now has $42.72 + 20x$.
Bob now has $42.72 + 15y$.
Carol now has $80 - 50(x + y)$.
We want all three to be equal. We first equate Alice and Bob to get
$20x = 15y$
$4x = 3y$
$y = \frac{4x}{3}$.
The KEY is that so long as $y = \frac{4x}{3}$, Alice and Bob are equal. Now plug $y = \frac{4x}{3}$ in the equality of Carol and Alice.

$$80 - 50(x + \frac{4x}{3}) = 42.72 + 20x.$$

$$37.28 - 50\frac{7x}{3} = 20x$$

$$27.28 = 116.66x + 20x$$

$$27.28 = 136.66x$$

$$x = 27.28/126.66 = 0.2153.$$

$$y = \frac{4x}{3} = 0.286$$

These x, y satisfy $x + y \leq 1$ so they work and make everyone equal.

9 Generalize This

We leave it to the reader to formalize the above procedure and generalize it to n people.

10 Properties

The n-player AW is equitable by design. It is Proportional by a similar proof that the 2-player AW is proportional. It is not Envy Free. In fact, we can show that there are cases where NO division will be Envy Free.