Number of Cuts
Exposition by William Gasarch

1 Introduction

Throughout this exposition (1) the term *protocol* means *proportional cake cutting protocol*, and (2) we only look at the worst case for number-of-cuts.

In the COME LATE protocol the number of cuts for 3 people is 5 (in all cases). In the TRIM protocol the number of cuts for 3 people is 3 (in the worst case). The DC protocol uses 3 cuts (in all cases).

Is there a protocol for 3 people that takes 2 cuts? We show that there is not. What about other numbers-of-people and number-of-cuts? In this exposition we show the following

1. For 3 people, 3 cuts are necessary and sufficient.
2. For 3 people, 4 cuts are necessary.
3. For n people, n cuts are necessary.
4. What about a bigger cake?
5. For 4 people, 4 cuts are sufficient.

2 $n = 3$: You Need Exactly 3 Cuts

Theorem 2.1

1. *There is a 3-person protocol that only uses 3 cuts. (This is the TRIM protocol so we do not prove it here.)*
2. *Any protocol for 3 people must use at least 3 cuts.*

Proof:

Assume, by way of contradiction, that there is a protocol for 3 people using 2 cuts. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone elses tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 1$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	1/2	1/2
Carol	1/2	1/2

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Alice and Bob's valuation as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	1/2	1/4	1/4
Carol	1/2	1/4	1/4

There are no more cuts to be made. Note that P_1 is the *only* piece acceptable to both Bob and Carol. They can't both have it! Hence the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = 1/2$.
- We assume $y_1 \leq y_2$, hence $y_1 \leq 1/4$ so P_{21} not acceptable to Bob.

WE set Alice and Carol's valuations as in the table below.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	1/2	$y_1 \leq$	y_2
Carol	1/2	1/4	1/4

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. But someone has to take it. Hence the protocol fails.
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3 An Attempt at a Lower Bound for $n = 4$

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We have that 3 people require 3 cuts. We want to prove that 4 people require 4 cuts. In this proof we can USE that 3 people require 3 cuts.

ATTEMPT AT PROOF THAT FOUR PEOPLE REQUIRE FOUR CUTS:

Assume, by way of contradiction, that there is a protocol where four people with just four cuts. Assume Alice has the first cut. We will set the valuations so that Bob, Carol, and Donna all thing one of the piece is BAD How bad? We'll say its worth $\frac{1}{4} - \epsilon$.

	P_1	P_2
Alice	x_1	x_2
Bob	$\frac{3}{4} + \epsilon$	$\frac{1}{4} - \epsilon$
Carol	$\frac{3}{4} + \epsilon$	$\frac{1}{4} - \epsilon$
Donna	$\frac{3}{4} + \epsilon$	$\frac{1}{4} - \epsilon$

So after one cut Alice can has to take P_2 and Bob, Carol, Donna have a protocol to split $\frac{3}{4} + \epsilon$ so that they each get 1/4. For concreteness lets take $\epsilon = \frac{1}{8}$. So now we have a 3-person protocol to split $\frac{7}{8}$ so that they each get $\frac{1}{4}$. If we scale this then we are saying that there is a 3-person protocol where they split a cake of size 1 and each get $\frac{1}{4}\frac{8}{7} = \frac{2}{7}$. We would have liked this to have been $\frac{1}{3}$ to get a contradiction.

Lets us revisit the $n = 3$ case. We want to say that even if they split a slightly bigger cake they can't all get 1/3.

4 $n = 3$ With a Wee Bit More Cake

We restate Theorem 2.1

Theorem 4.1 *Any protocol for 3 people that starts off with a cake of size 1, and guarantees that everyone gets* $\geq \frac{1}{3}$*, uses at least 3 cuts.*

What if we started out with JUST a bit more cake?

Theorem 4.2 *Let* $0 \leq \epsilon < \frac{1}{3}$*. Any protocol for 3 people that starts off with a cake of size* $1 + \epsilon$*, and guarantees that everyone gets* $\geq \frac{1}{3}$*, uses at least 3 cuts. (Note that all 3 people value the entire cake at* $1 + \epsilon$*.)*

Proof:

Assume, by way of contradiction, that there is a protocol for 3 people using 2 cuts that splits a cake of size $1+\epsilon$ into three pieces so that each person gets a piece of size $\geq 1/3$. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone elses tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1+x_2 = 1+\epsilon$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	$(1+\epsilon)/2$	$(1+\epsilon)/2$
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/2$

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Bob and Carol's valuations as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$(1+\epsilon)/4$
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$(1+\epsilon)/4$

There are no more cuts to be made.

We WANT to make P_{21} and P_{22} NOT acceptable to Bob or Carol. Hence we need

$$\begin{aligned}(1+\epsilon)/4 &< 1/3 \\ 1+\epsilon &< 4/3 \\ \epsilon &< 1/3\end{aligned}$$

NOW note that P_1 is the *only* piece acceptable to both Bob and Carol. They can't both have it! Hence the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = (1+\epsilon)/2$.

- We assume $y_1 \leq y_2$ hence $y_1 \leq (1+\epsilon)/4 < 1/3$ so P_{22} is not acceptable to Bob.

WE set Alice and Carol's values by the following table.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	$(1_+\epsilon)/2$	$y_1 \leq$	y_2
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$1/4$

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. But someone has to take it. Hence the protocol fails.
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5 A Different Viewpoint

We introduce a useful notation and restate a scaled version Theorem 4.2 in that notation.

Def 5.1 Let s, p be positive rationals. Let $n \in \mathsf{N}$. An (n, s, p) *protocol* is a protocol for n people which takes a cake of size s and gives everyone $\geq p$.

Note 5.2 An $(n, 1, 1/n)$ protocol is an n person proportional protocol.

Theorem 5.3 *For all y, for all $\epsilon > 0$, there is no 2-cut $(3, 4y-\epsilon, y)$ protocol.*

Proof: Assume, by way of contradiction, that there is a 2-cut $(3, 4y-\epsilon, y)$ protocol. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone elses tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 4y + \epsilon$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	$2y - (\epsilon/2)$	$2y - (\epsilon/2)$
Carol	$2y - (\epsilon/2)$	$2y - (\epsilon/2)$

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Bob and Carol's valuations as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$
Carol	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$

There are no more cuts to be made.

Since neither P_{21} nor P_{22} are acceptable to Bob or Carol they both get P_1. They can't both have it! So the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = 2y - (\epsilon/2)$.
- We assume $y_1 \le y_2$ hence $y_1 \le y - (\epsilon/4)$ so P_{22} is not acceptable to Bob.

WE set Alice and Carol's values by the following table.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	$2y - (\epsilon/2)$	$y_1 \le$	y_2
Carol	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. Hence the protocol fails.

▮

Corollary 5.4 *There is no 3-person 2-cut proportional protocol.*

Proof: If we plug $y = 1/3$ and $\epsilon = 1/3$ into Theorem 5.3 we obtain that there is no 2-cut $(3, 1, 1/3)$ protocol. ▮

6 $n = 4$: You Need at Least Four Cuts

We want to show that there is no 4-person 3-cut prop. protocol. We show something stronger which we will then use in our 5-person lower bound.

Theorem 6.1 *For all y, for all $\epsilon > 0$, there is no 3-cut $(4, 5y-\epsilon, y)$ protocol.*

Proof:

Assume, by way of contradiction, that there is a 3-cut $(4, 5y - \epsilon, y)$ protocol. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone elses tastes.

Note that after 3 cuts there will be exactly 4 pieces. Hence if there is a piece that only one player likes, that player must get it.

The players are Alice, Bob, Carol, and Donna. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 5y - \epsilon$. WE set Bob and Carol as in the table below (we determine δ later).

	P_1	P_2
Alice	x_1	x_2
Bob	$4y - (\epsilon/2)$	$y - (\epsilon/2)$
Carol	$4y - (\epsilon/2)$	$y - (\epsilon/2)$
Donna	$4y - (\epsilon/2)$	$y - (\epsilon/2)$

Alice is the only one who likes P_2. Hence Alice will get P_2. Consider the rest of the protocol. It is a 3-person protocol with Bob, Carol, Donna

They are splitting a cake of size $4y - (\epsilon/2)$.

They are each getting y.

They are only using 2 cuts.

Hence they have a 2-cut $(3, 4y - (\epsilon/2), y)$ protocol.

This contradicts Theorem 5.3 (with $(\epsilon/2)$ instead of ϵ— recall that Theorem 5.3 held for ALL ϵ.)

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Corollary 6.2 *There is no 4 person 3-cut prop. protocol.*

Proof: If we plug $y = 1/4$ and $\epsilon = 1/4$ into Theorem 6.1 we obtain that there is no 3-cut $(4, 1, 1/4)$ protocol. ∎

We leave it to the reader to show, from Theorem 6.1 that for all y, for all $\epsilon > 0$, there is no 4-cut $(5, 6y - \epsilon, y)$ protocol.

7 Lower Bounds For General n

Theorem 7.1 *For all n, for all y, for all $\epsilon > 0$, there is no $(n-1)$-cut $(n, (n+1)y - \epsilon, y)$ protocol.*

Proof: We prove this by induction.

The base case is $n = 3$, which is Theorem

We show that if the theorem is true for $n-1$ then it is true for n. SO- we ASSUME

STATEMENT I:

for all y, for all $\epsilon > 0$, there is no $(n-2)$-cut $(n-1, ny - \epsilon, y)$ protocol.

We try to prove *for all y, for all $\epsilon > 0$, there is no $(n-1)$-cut $(n, (n+1)y - \epsilon, y)$ protocol.*

Assume, by way of contradiction, that

there exists y, there exists $\epsilon > 0$, there is an $(n-1)$-cut $(n, (n+1)y - \epsilon, y)$ protocol.

The players are $A_1, \ldots, A_n$. We can assume that A_n makes the first cut. WE will set the opinions of $A_1, \ldots, A_n$.

	P_1	P_2
A_1	$ny - (\epsilon/2)$	$y - (\epsilon/2)$
A_2	$ny - (\epsilon/2)$	$y - (\epsilon/2)$
A_3	$ny - (\epsilon/2)$	$y - (\epsilon/2)$
$\vdots$	$ny - (\epsilon/2)$	$y - (\epsilon/2)$
A_{n-1}	$ny - (\epsilon/2)$	$y - (\epsilon/2)$
A_n	x_1	x_2

After this there is an $n-2$-cut protocol for the n players $A_1, \ldots, A_n$, where the cake is of size $ny - (\epsilon/2)$ where each player gets at least y. So they have an $n-2$-cut protocol for $(n-1, ny - (\epsilon/2), y)$. This contradicts statement I above. ∎

Corollary 7.2 *There is no n person $(n-1)$-cut prop. protocol.*

Proof: If we plug $y = 1/n$ and $\epsilon = 1/n$ into Theorem 7.1 we obtain that there is no $(n-1)$-cut $(n, 1, 1/n)$ protocol. ∎

8 How Big a Cake Do we Need for $n = 3$ case and 2 cuts?

Lets look at the case of n people and $y = 1/n$.

For all n, for all $\epsilon > 0$, there is no $(n-1)$-cut $(n, 1 + \frac{1}{n} - \epsilon, \frac{1}{n})$ protocol.

Lets look at the $n = 3$ case:

For all $\epsilon > 0$, there is no 2-cut $(3, 1 + \frac{1}{3} - \epsilon, \frac{1}{3})$ protocol.

This raises the obvious question:

Is there a 2-cut $(3, 1 + \frac{1}{3}, \frac{1}{3})$ protocol?

YES! Before proceeding we introduce a notation that will make the protocols easier: All of the protocols will begin with Alice cutting the cake in half with the advice to make it even. The pieces will be called P_1, P_2 We will then refer to how the rest of the people *split* meaning the number who prefer P_1 and the number who prefer P_2. We will then say WHO prefers which but this is of course arbitrary. We regard an $a - b$ split and a $b - a$ split as equivalent so we only look at one of those. We will always take the first a alphetically to be on the a-side.

Also- note that cut-and-choose can be viewed as a 1-cut $(2, 2y, y)$ protocol. we will call it that for consistency.

2-cut protocol for $(3, 4/3, 1/3)$.

1. Alice cuts the cake (in half- so 2/3 and 2/3) We call the pieces P_1, P_2.
2. If Bob and Carol split 2-0 then Bob and Carol do $(2, 2/3, 1/3)$ on P_1, Alice takes P_2. Note that Bob and Carol get $\geq 1/3$ and Alice gets 2/3.
3. If Bob and Carol split 1-1 then Alice and Carol do $(2, 2/3, 1/2)$ on P_1, Bob takes P_2. Note that Alice and Carol get $\geq 1/3$ and Bob gets $\geq 2/3$.

What about $n = 4$? General n?

We are better off if we first generalize our result for $n = 3$.

Theorem 8.1 *For all y there is a 2-cut $(3, 4y, y)$ protocol.*

Proof:

1. Alice cuts the cake (in half- so $2y$ and $2y$) We call the pieces P_1, P_2.
2. If Bob and Carol split 2-0 then Bob and Carol do $(2, 2y, y)$ P_1, Alice takes P_2. Note that Bob and Carol get $\geq y$ and Alice gets $2y$.

3. If Bob and Carol split 1-1 then Alice and Carol do $(2, 2y, y)$ on P_1, Bob takes P_2. Note that all get $\geq y$.

▮

9 How Big a Cake Do we Need for $n = 4$ case and 3 cuts?

Theorem 9.1 *For all* y *there is a 3-cut* $(4, 6y, y)$ *protocol.*

Proof:

1. Alice cuts the cake (in half- so $3y$ and $3y$) We call the pieces P_1, P_2.
2.
3. **Case 1:** Bob, Carol, Donna split 2-1. Bob and Carol do $(2, 3y, y)$ on P_1 (so they each get $\geq 1.5y$). Alice and Donna do $(2, 3y, y)$ P_2 (so they each get $\geq 1.5y$).
4. **Case 2:**(The hard case) Bob, Carol, Donna split 3-0. Bob, Carol, Donna all think $P_1 \geq 3y$ and $P_2 \leq 3y$.
5. **Case 2.1:** Bob (or Carol or Donna) thinks $P_2 \geq 2y$. Alice and Bob do $(2, 2y, y)$ on P_2 (Alice thinks $P_2 = 2y$ so Bob gets $\geq y$ and Alice gets $\geq 1.5y$) Carol and Donna do $(2, 3y, y)$ (each gets $\geq 1.5y$).
6. **Case 2.2:** Bob, Carol, and Donna ALL think $P_2 < 2y$. Hence they all think $P_1 \geq 4y$. Alice gets P_2. Bob, Carol, and Donna do the $(3, 4y, y)$ protocol on P_1.

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Corollary 9.2 *There is a 3-cut* $(4, \frac{3}{2}, \frac{1}{4})$ *protocol.*

Note the following contrast:

- There is NO 3-cut $(4, \frac{4}{3} - \epsilon, \frac{1}{4})$ protocol.
- There is a 3-cut $(4, \frac{3}{2}, \frac{1}{4})$ protocol.

We have an open question (at least to us) here. Find x such that

- There is NO 3-cut $(4, x-\epsilon, \frac{1}{4})$ protocol.
- There is a 3-cut $(4, x, \frac{1}{4})$ protocol.

10 How Big a Cake Do we Need for $n = 5$ case and 4 cuts?

Lets look at the last two theorems:

For all y there is a 2-cut $(3, 4y, y)$ protocol.
For all y there is a 3-cut $(4, 6y, y)$ protocol.
Dare we guess that there is a 4-cut $(5, 8y, y)$-protocol?

Theorem 10.1 *For all y there is a 4-cut $(5, 8y, y)$ protocol.*

Proof:

1. Alice cuts the cake (in half- so $4y$ and $4y$) We call the pieces P_1, P_2.
2. **Case 1:** Bob, Carol, Donna, Edgar split either 1-3 or 2-2. Adding Alice to the 1-side of 1-3 or any side of 2-2 you have a 2-3 split for all of the people. Lets say Alice and Bob think $P_1 \geq 4y$ and Carol, Donna, Edgar think $P_2 \geq 4y$. Alice and Bob do $(2, 4y, y)$ Carol, Donna, Edgar do $(3, 4y, y)$ protocol.
3. **Case 2:**(The hard case) If Bob, Carol, Donna, Edgar split 4-0. Bob, Carol, Donna, Edgar all think $P_1 \geq 4y$ and $P_2 \leq 4y$. KEY: We order how much they like P_1 by Bob, Carol, Donna, Edgar. Hence Bob likes P_1 the least (though still at least $4y$).
4. **Case 2.1:** Bob thinks $P_1 \leq 6y$. Hence Bob thinks $P_2 \geq 2y$. Alice and Bob do $(2, 2y, y)$ with P_2 (Alice thinks $P_2 \geq 4y$, so Bob gets $\geq y$ and Alice gets $\geq 2y$). Carol and Donna and Edgar do the $(3, 4y, y)$ protocol.
5. **Case 2.2:** Bob thinks $P_1 > 6y$. Since Bob had the lowest opinion of P_1, they ALL think $P_1 \geq 6y$. Hence Bob, Carol, Donna, Edgar can use the $(4, 6y, y)$-protocol on P_1, and they each get y. Alice takes P_2 and gets $4y$.

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Corollary 10.2 *There is a 4-cut* $(5, \frac{8}{5}, \frac{1}{5})$ *protocol.*