Aspects of Fairness
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1 Introduction

We discuss different definitions of fairness and other qualities you might want in a protocol or division.

2 Fairness

There are n people $A_1, \ldots, A_n$. They are going to split an item (or set of items). They all think the item is worth 1. Person A_i has a valuation of V_i what certain parts of the item are worth. If the item a cake modeled as the interval $[0,1]$ then V_i would map the entire interval to 1 and subintervals to numbers ≤ 1 in a reasonable way. If item is a set of items (Picasso, Car) then V_i might map Picasso to $\frac{3}{4}$ and Car to $\frac{1}{4}$.

Def 2.1 A division is $(P_1, \ldots, P_n)$ where A_i gets P_i.

1. A division is *Proportional* if $(\forall i)[V_i(P_i) \geq \frac{1}{n}]$. So everyone thinks they got at least $\frac{1}{n}$.
2. A division is *Envy-Free* if $(\forall i, j)[V_i(P_i) \geq V_i(P_j)]$. So everyone things they have the biggest piece (or tied).
3. A division is *Equitable* if $(\forall i, j)[V_i(P_i) = V_j(P_j)]$. So everyone got the exact same size piece.

Def 2.2 An division $(P_1, \ldots, P_n)$ is *better than* a division $(P'_1, \ldots, P'_n)$ if some player does better (that is, there is an i such that $V_i(P'_i) > V_i(P_i)$ and no player does worse.

Def 2.3 A division is *Efficient* if there is no better division. A protocol is *efficient* if it always results in an efficient division.

3 Protocols

Our protocols will have two aspects to them.

There is the demands on the individuals that all players can verify. Examples:

1. Alice cuts the pie into two pieces.
2. Alice allocates to each item a value with the sum being 100.

There is advice that we give the player but he or she need not follow. Examples:

1. Alice cuts the pie into two *equal* pieces.
2. Alice allocates to each item how much its worth with the sum being 100 and does it honestly.

The following terms are not standard but they are implicit in the literature.

Def 3.1

1. If a player follows the advice then we say he is *honest* or *played honestly*. If a player does not follow the advice then we say he *cheated*.
2. A protocol is *cheat proof* if for all ways that a player can cheat, there is a scenario where he will get LESS than if he played honestly.
3. A protocol is *super cheat proof* if even if a player knows the others players preferences, cheating may lead to him getting less than if he was honest.
4. A protocol is *proportional cheat proof* if even if a player cheats, the rest get $\geq \frac{1}{n}$.
5. A protocol is *envy-free cheat proof* if even if a player cheats, the rest all think they got the biggest piece (or tied).