Envy-Free Discrete Protocols PROJECT
Morally DUE April 21. Can hand in Imorally April 23

This project is in two parts:

1. Present the $n = 5$ envyfree discrete protocol. It must be so clear that someone who has not seen the $n = 4$ protocol can understand it. You can use my notes as a template. I have made them available on the website so you can downland text and edit it.
2. Present the envyfree discrete protocol for general n. It must be so clear that someone who has not seen the $n = 4$ protocol can understand it. You can use my notes as a template. I have made them available on the website so you can

Below I have done, for the $n = 5$ case, made clear what I expect of you.

1 An all-but-ϵ Envy-Free Protocol for 5 People

Def 1.1 An *all-but-ϵ envy-free protocol* divides the cake, *except* a piece that everyone agrees is $\leq \epsilon$, in an envy-free way. The piece of size $\leq \epsilon$ is not allocated.

Theorem 1.2 *For every $\epsilon > 0$ there is an all-but-ϵ envy-free protocol for 5 people.*

Proof:
FIRST-PERSON-HAPPY(A,B,C,D,E;P)
YOU FILL IN. YOU DO NOT NEED TO PROVE THAT IT WORKS BUT IT HAS TO WORK.
END OF FIRST-PERSON-HAPPY
ALL-HAPPY(A,B,C,D;P)
YOU FILL IN. YOU DO NOT NEED TO PROVE THAT IT WORKS BUT IT HAS TO WORK.
END OF ALL-HAPPY(A,B,C,D,E;P)
ALMOST-ENVY-FREE5$(A, B, C, D, E; P; \epsilon)$

YOU FILL IN. YOU DO NOT NEED TO PROVE THAT IT WORKS BUT IT HAS TO WORK. **END OF ALMOST-ENVY-FREE4**

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1.1 Making Alice and Bob REALLY Disagree

THE FOLLOING LEMMA YOU CAN USE AND NOT PROVE. WE PROVED IT IN CLASS.

Lemma 1.3 *Assume that Alice and Bob are both looking at pieces P, Q and Alice things $P = Q$ while Bob thinks $P > Q$. There is a protocol that produces P', Q' such that*

- *Alice thinks $P' < Q'$.*
- *Bob thinks $P' > Q'$.*
- *$P \cup Q = P' \cup Q'$.*

1.2 Making Alice and Bob Have an Advantage Over Each Other

Lemma 1.4 *Assume that Alice and Bob are both looking at pieces P, Q and Alice thinks $P = Q$ while Bob thinks $P > Q$. There is a protocol that produces (likely very small) pieces $p_1, \ldots, p_k, q_1, \ldots, q_k$ such that YOU NEED TO FILL IN k SUCH THAT WHAT YOU PROVE IS USEFUL FOR THE NEXT LEMMA. YOU MAY WANT TO TRY TO DO THE NEXT LEMMA FIRST.*

- *Alice thinks $q_1 = \cdots = q_k > p_1, \ldots, p_k$.*
- *Bob thinks $p_1 = \cdots = p_k > q_1, \ldots, q_k$.*

Proof:
$2k$ **FUNKY PIECES**

1. Alice and Bob run the protocol from Lemma 1.3 to obtain P', Q' with $P' < Q'$ and $P' > Q'$. For notation we rename them and assume Alice thinks $P < Q$ and Bob thinks $P > Q$.

2. Bob names a number $m \geq X$ (YOU NEED TO FILL IN X) (m should be big enough so that no matter how P is cut into m pieces, if Bob discards the Y smallest pieces, then he still thinks he has more than Alice. YOU NEED TO FILL IN Y. WARNING DO NOT JUST GUESS AND HOPE IT WORKS. DO THE PROOF AND SEE WHAT Y WORKS.

3. Alice cuts P into m pieces $P_1, \ldots, P_m$ and Q into m pieces $Q_1, \ldots, Q_m$. (Cuts them both equally. Note that Alice will think

$$P_1 = \cdots = P_m < Q_1 = \cdots = Q_m.$$

)

4. Bob sorts the pieces:

   (a) Bob thinks that $P_m \leq P_{m-1} \leq \cdots \leq P_3 \leq P_2 \leq P_1$.

   (b) Bob thinks that $Q_m \leq Q_{m-1} \leq \cdots \leq Q_3 \leq Q_2 \leq Q_1$.

5. If Bob thinks $P_k > Q_{m-(k-1)}$ then

   (a) Bob trims $P_1, \ldots, P_k$ (down to P_k value). Let the trimmed versions be $P_1' \ldots, P_k'$. Let $p_1 = P_1'$, …, $p_k = P_k'$.

   (b) Let $q_1 = Q_{m-k-1}$, $q_2 = Q_{m-1}$, $q_3 = Q_m$.

   Why this works:

   - Bob thinks $p_1 = \cdots = p_k$ and, since he thinks $P_k > Q_{m-k-1}$ he thinks

     $$p_1 = \cdots = p_k = P_k > \{Q_{m-k-1} \geq Q_{m-1} \geq Q_m\} = \{q_1 \leq \cdots \leq q_k\}.$$

   - Alice thinks

     $$Q_m = q_1 = \cdots = q_k > P_1 \geq P_1' = p_1 = \cdots = p_k.$$

6. If Bob thinks $P_k \leq Q_{m-k-1}$ then what do we do? INTUITION: Since Bob thinks

   (1) $P_k \leq Q_{m-k-1}$,

(2) the union of all of the P_i's is more than the union of all of the Q_i's,

(3) P_k is the kth largest P_i,

Bob must think P_1 is really big!.

(a) Bob cuts P_1 into k (equal) pieces. Call them $p_1, \ldots, p_3$.

(b) Let $q_1 = Q_m$, …, $q_k = Q_{m-k-1}$.

We show later why this works.

The only case we didn't prove works was the last one. For Alice this is easy. She thinks

$$q_1 = \cdots = q_k > P_1 \geq p_1, \ldots, p_k.$$

What about Bob? This is more complicated. YOU NEED TO FINISH THIS PROOF. THIS WILL LEAD YOU TO THE CORRECT VALUE OF m EARLIER. ▮

1.3 Making Alice and Bob Have an Advantage Over Each Other

We can now use this Theorem 1.2 and Lemma 1.4 to obtain a protocol where Alice and Bob have an advantage over each other!

Theorem 1.5 *Alice, Bob, Carol, Donna, and Edgar are looking at 3 pieces P, Q, R (any of these could themselves be composed of many pieces). Alice thinks $P = Q$. Bob thinks $P > Q$. There is a protocol such that at the end:*

- *all but some trim T of the cake has been divided*
- *on the part that was divided the division is Envy Free*
- *Alice has an advantage over Bob*
- *Bob has an advantage over Alice.*

YOU NEED TO PROVE THIS ALL BY YOURSELF. YOU WILL USE THE PRIOR LEMMA

1.4 The Final Envy Free Protocol for 5 People

YOU MUST DO THIS ALL ON YOUR OWN.