Unequal Division

William Gasarch-U of MD

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Discuss how they can do this.

A Terrible 5 : 8 Protocol

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Why Terrible Lots of cloning and lots of cuts (I think 35 cuts). Should number-of-clones even be a complexity measure? Ethical question: When you fuse them all into A (or B) is that murder?

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CON Number of cuts is 12. Surely we can do better than that.
Response Indeed we can, and don't call me Shirley.

There are Better Protocols But

There are better protocols but they are better on blackboard
So for now these slides are at an end.

Protocols Bill Does on Blackboard

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(8:5) with 5 cuts

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$(a : b)$ - see next slide for two approaches.

Two Approaches

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For $(a : b)$ with $a < b$.

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If not then A takes it.

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Answer on next slide

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The paper is here: `https://arxiv.org/abs/1206.1553`

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4) Pick n, m, L, k_1, k_2. Hope $2^n(2^n + 2k_1)$,$3^m(3^m + 2k_2)$ rel prime.

Putting it all Together

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To make the first number big set $k_1 = 6$ (a guess).

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Next slide for L.

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$783 - L \equiv 0 \pmod{16}$

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Use the Chinese Remainder Theorem to derive that $L = 367$.

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$L \equiv 448 \equiv 16 \pmod{27}$.

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Note that $L < 448$ and $L < 783$ so this will work.

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A Cuts [367 : 864].

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A Cuts $[367 : 864]$.

If B likes 367 piece then problem is now
$(448 : 783 - 367) = (448 : 416) = (14 : 13)$.

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If not then A takes 367 piece and the problem is now
$(448 - 367 : 864) = (81 : 864) = (3 : 32)$.

A General Procedure to Find Good $(a : b)$

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3) Use Chinese Remainder Theorem to solve

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3) Use Chinese Remainder Theorem to solve
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5) Even if works, check it anyway and see how m any cuts it saves.