

Answers to Worksheet on Definitions of Fairness

William Gasarch-U of MD

Problem One

	l_1	l_2	l_3	l_4	l_5	l_6
A						
B						
C						

Fill in the above with natural numbers such that the following hold.

- ▶ The numbers in a row add up to 12.
- ▶ There is an allocation where everyone gets 12 and nobody is envious.

Answer to Problem One

	l_1	l_2	l_3	l_4	l_5	l_6
A	6	6				
B			6	6		
C					6	6

- ▶ The numbers in a row add up to 12.
- ▶ Everyone gets 12 and nobody is envious.

Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A						
B						
C						

Fill in the above with natural numbers such that the following hold.

- ▶ The numbers in a row add up to 12.
- ▶ There is an allocation where
 - ▶ Everyone gets ≥ 4 (so its proportional).
 - ▶ A envies B , B envies C , C envies A , no other envy.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A's view: A gets 4.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0. B gets 4.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0. B gets 4. C gets 10.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0. B gets 4. C gets 10.

C 's view: A gets 10.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0. B gets 4. C gets 10.

C 's view: A gets 10. B gets 0.

Answer to Problem Two

	l_1	l_2	l_3	l_4	l_5	l_6
A	2	2	5	5	0	0
B	0	0	2	2	5	5
C	5	5	0	0	2	2

A 's view: A gets 4. B gets 10. C gets 0.

B 's view: A gets 0. B gets 4. C gets 10.

C 's view: A gets 10. B gets 0. C gets 4.

Problem Three

Show that there is a valuation and allocation for 10 people $A_1, \dots, A_{10}$ such that

- ▶ YOU get to pick the number of items and the row-totals.
- ▶ The allocation is proportional.
- ▶ A_1 envies A_2 , A_2 envies A_3 , $\dots$, A_9 envies A_{10} , A_{10} envies A_{11} .
- ▶ There is no other envy.

Answer to Problem Three

S and L to be picked later.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L								
A_2		S	L							
A_3			S	L						
A_4				S	L					
A_5					S	L				
A_6						S	L			
A_7							S	L		
A_8								S	L	
A_9									S	L
A_{10}	L									S

Answer to Problem Three

S and L to be picked later.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L								
A_2		S	L							
A_3			S	L						
A_4				S	L					
A_5					S	L				
A_6						S	L			
A_7							S	L		
A_8								S	L	
A_9									S	L
A_{10}	L									S

Need (1) $S \geq \frac{S+L}{10}$ for proportionality

Answer to Problem Three

S and L to be picked later.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L								
A_2		S	L							
A_3			S	L						
A_4				S	L					
A_5					S	L				
A_6						S	L			
A_7							S	L		
A_8								S	L	
A_9									S	L
A_{10}	L									S

Need (1) $S \geq \frac{S+L}{10}$ for proportionality

Need (2) $S < L$ for envy.

Answer to Problem Three

S and L to be picked later.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L								
A_2		S	L							
A_3			S	L						
A_4				S	L					
A_5					S	L				
A_6						S	L			
A_7							S	L		
A_8								S	L	
A_9									S	L
A_{10}	L									S

Need (1) $S \geq \frac{S+L}{10}$ for proportionality

Need (2) $S < L$ for envy.

From (1) we get $10S \geq S + L$, so $9S \geq L$. So need $\frac{L}{9} \leq S < L$.

Answer to Problem Three

S and L to be picked later.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L								
A_2		S	L							
A_3			S	L						
A_4				S	L					
A_5					S	L				
A_6						S	L			
A_7							S	L		
A_8								S	L	
A_9									S	L
A_{10}	L									S

Need (1) $S \geq \frac{S+L}{10}$ for proportionality

Need (2) $S < L$ for envy.

From (1) we get $10S \geq S + L$, so $9S \geq L$. So need $\frac{L}{9} \leq S < L$.

There are many solutions. Here is one: $L = 18$, $2 \leq S < 18$.

Problem Four

Show that there is a valuation and allocation for 10 people $A_1, \dots, A_{10}$ such that

- ▶ YOU get to pick the number of items and the row-totals.
- ▶ The allocation is proportional.
- ▶ Everyone envies everyone.
- ▶ You DO NOT have to use natural numbers.

Confession My approach did not work. I suspect this problem is impossible. I go through my approach and why it does not work.

Answer to Problem Four

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L	L	L	L	L	L	L	L	L
A_2	L	S	L	L	L	L	L	L	L	L
A_3	L	L	S	L	L	L	L	L	L	L
A_4	L	L	L	S	L	L	L	L	L	L
A_5	L	L	L	L	S	L	L	L	L	L
A_6	L	L	L	L	L	S	L	L	L	L
A_7	L	L	L	L	L	L	S	L	L	L
A_8	L	L	L	L	L	L	L	S	L	L
A_9	L	L	L	L	L	L	L	L	S	L
A_{10}	L	L	L	L	L	L	L	L	L	S

Answer to Problem Four

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L	L	L	L	L	L	L	L	L
A_2	L	S	L	L	L	L	L	L	L	L
A_3	L	L	S	L	L	L	L	L	L	L
A_4	L	L	L	S	L	L	L	L	L	L
A_5	L	L	L	L	S	L	L	L	L	L
A_6	L	L	L	L	L	S	L	L	L	L
A_7	L	L	L	L	L	L	S	L	L	L
A_8	L	L	L	L	L	L	L	S	L	L
A_9	L	L	L	L	L	L	L	L	S	L
A_{10}	L	L	L	L	L	L	L	L	L	S

Need (1) $S \geq \frac{S+9L}{10}$ for prop. So $10S \geq S + 9L$, so $S \geq L$.

Answer to Problem Four

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}
A_1	S	L	L	L	L	L	L	L	L	L
A_2	L	S	L	L	L	L	L	L	L	L
A_3	L	L	S	L	L	L	L	L	L	L
A_4	L	L	L	S	L	L	L	L	L	L
A_5	L	L	L	L	S	L	L	L	L	L
A_6	L	L	L	L	L	S	L	L	L	L
A_7	L	L	L	L	L	L	S	L	L	L
A_8	L	L	L	L	L	L	L	S	L	L
A_9	L	L	L	L	L	L	L	L	S	L
A_{10}	L	L	L	L	L	L	L	L	L	S

Need (1) $S \geq \frac{S+9L}{10}$ for prop. So $10S \geq S + 9L$, so $S \geq L$.

Need (2) $S < L$ for Envy. Can't get both.