Homework 3, MORALLY Due Feb 17

1. (0 points but you have to answer) What is your name? Write it clearly. Staple your HW.

2. (10 points) In Led Zeppelin song *Stairway to Heaven* they say that profound line **All that Glitters is Not Gold**

   (a) What does this mean literally?

   (b) Write a quantified statement that captures what they intended.

   Literally it means IF x glitters THEN x is NOT GOLD. We write this as

   $$(\forall x)[GLIT(x) \implies \neg GOLD(x)]$$

   where $GLIT(x)$ means that x Glittes, and $GOLD(x)$ means that x is Gold. The domain is just things.

   Of course we note that since if x is a gold nugget then $GLIT(x)$ is true AND $GOLD(x)$ is true. Hence the statement is not true and is absurd.

   What Led Zeppline MEANT to say is that just because something glitters it is not gold. Don't be fooled. This translates to the much less poetic

   $$(\exists x)[GLIT(x) \wedge \neg GOLD(x)]$$

3. (10 points) Abe Lincoln said **You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time.**

   Define domains and predicates so that you can express this statement and then express it in terms of quantifiers.

   t ranges over TIME, p ranges over PEOPLE. This defines the domains- if you see at t in a quantifier it is ranging over time, if you see a p it is ranging over people.

   $FOOL(p, t)$ means person p is fooled at time t.

Our statmenet will be the AND of the following statements. We do this carefully.

You can fool all of the people some of the time

So there is some time where you can fool ALL of the people:

$$(\exists t)(\forall p)[FOOL(p, t)]$$

you can fool some of the people all of the time

So there are some people you can fool all of the time.

$$(\exists p)(\forall t)[FOOL(x, t)]$$

you can't fool all of the people all of the time.

$$\neg(\forall p)(\forall t)[FOOL(x, t)]$$

4. (20 points) Write the following in terms of quantifiers: **There is exactly one number that requires 9 cubes to sum to it– All of the rest can be done with 8.**

Note that we are saying there is ONLY one such number.

We first build up some predicates.

$NINECUBE(x)$ we define as

$$(\exists y_1, y_2, y_3, y_4.y_5.y_6, y_7, y_8, y_9)[x = y_1^3+y_2^3+y_3^3+y_4^3+y_5^3+y_6^3+y_7^3+y_8^3+y_9^3].$$

$EIGHTCUBE(x)$ we define as

$$(\exists y_1, y_2, y_3, y_4.y_5.y_6, y_7, y_8)[x = y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 + y_6^3 + y_7^3 + y_8^3].$$

We use these in our expression:

$$(\exists x)[NINECUBE(x) \wedge \neg EIGHTCUBE(x) \wedge (\forall y)[y \neq x \rightarrow EIGHTCUBE(y)]]$$

5. (20 points) Let D be a domain and $P(x)$ be a predicate over D. Write the following in terms of quantifiers: **There are exactly 5 elements of the domain for which P is true.**

   $(\exists x_1, x_2, x_3, x_4, x_5)$

   [

   $x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_1 \neq x_5 \wedge$

   $x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_2 \neq x_5 \wedge$

   $x_3 \neq x_4 \wedge x_3 \neq x_5 \wedge x_4 \neq x_5 \wedge$

   $P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4) \wedge P(x_5) \wedge$

   $(\forall y)[P(y) \rightarrow (y = x_1 \vee y = x_2 \vee y = x_3 \vee y = x_4 \vee y = x_5)]$

   ]

6. (30 points) For each of the following sentences

   - Find a finite but nonempty domain where it is true OR prove that there is no such.
   - Find an infinite domain where it is true OR prove there is no such.

   (a) $(\forall x)(\forall y)(\exists z)[x < y \implies x < z < y]$

   Finite nonempty domain where it is true: $\{0\}$. Note that since there is only one element of the domain you never have $x < y$. Hence this is true VACUOUSLY- there is no (x, y) with $x < y$ to apply it to. This is a cheap trick.

   If I had asked for a domain of size ≥ 2 then there is no such. Proof: Let x, y such that $x < y$ be in the domain. Then there is some z with $x < z < y$. Now there are three elements. Now take x and z in premise. There is an element between x and z. Keep doing this and you will get that there is no finite bound on the number of elements in D.

   Infinite domain: The rationals.

   (b) $(\forall x)(\exists y)[y^2 = x]$.

   Finite domain where this is true: $\{0, 1\}$ If $x = 0$ then $y = 0$ works. If $x = 1$ then $y = 1$ works. (Any domain with at least three elements has to have an infinite number of elements.)

   Infinite domain where it is true: The Positive Reals.

(c) $(\forall x)(\exists y)[x \leq y]$.

Finite domain where it is true: Any finite set works since you can always take $y = x$.

Infinite domain where it is true: Any infinite set works since you can always take $y = x$.

(d) $(\forall x)(\exists y)[x < y]$.

Finite domain where it is true: There aren't any. If x_1 is in the domain then so is some number x_2 with $x_1 < x_2$. Then there is a number x_3 with $x_2 < x_3$. Etc.

Infinite domain where it is true: the natural numbers. Take $y = x + 1$.

(e) $(\exists y)(\forall x)[x < y]$.

This is not true in any domain since if you fix$y$ then $x < y$ is not always true since you can take $x = y$.

(f) $(\exists x)(\exists y)(\forall z)[x < y \wedge x \leq z \leq y]$.

Finite set: any finite set with at least two elements works since it will have a max and min element that are different from each other.

Infinite set: take $[0, 1]$, the set of all reals between 0 and 1 but INCLUDING 0 and 1.