Homework 7 Morally Due Mar 31
WARNING- THIS HW IS TWO PAGES LONG.

1. (0 points but you have to answer) What is your name? Write it clearly. Staple your HW. When is midterm 2?

2. (20 points) Prove by induction that for all $n \ge 1$

$$\sum_{i=1}^{n} i2^i = 2 + (n-1)2^{n+1}$$

SOLUTION TO PROBLEM 2

Base Case: $n = 1$. $1 \times 2^1 = 2 + (1-1)2^2$ is $2 = 2$.

Induction Hypothesis: We can assume

$$\sum_{i=1}^{n-1} i2^i = 2 + (n-2)2^n$$

Induction Step:

Add $n2^n$ to both sides.

$$(\sum_{i=1}^{n-1} i2^i) + n2^n = 2 + (n-2)2^n + n2^n$$

$$\sum_{i=1}^{n} i2^i = 2 + (n-2+n)2^n = 2 + (2n-2)2^n = 2 + (n-1)2^{n+1}$$

END OF SOLUTION TO PROBLEM 2

3. (20 points) Let P be a statement about the integers. Assume we know the following:

$P(4)$

$(\forall n \in \mathsf{Z})[P(n) \implies P(n+3)]$.

$(\forall n \in \mathsf{Z})[P(n) \implies P(n-3)]$.

For which $n \in \mathsf{Z}$ do we know that $P(n)$ holds?

SOLUTION TO PROBLEM 3

Since we have $P(4)$ and $P(n) \implies P(n+3)$ we get P on $\{4, 7, 10, 13, \ldots\}$

Since we have $P(4)$ and $P(n) \implies P(n-3)$ we get P on $\{4, 1, -2, -5, \ldots\}$

Note that all of these numbers are $\equiv 1 \pmod 3$. So we have $P(n)$ for all n in the set

$$\{n : n \equiv 1 \pmod 3\}.$$

Notice that, in general, repeatedly adding k to or subtracting k from a number a generates all the numbers that are equivalent to $a \mod k$

END OF SOLUTION TO PROBLEM 3.

4. (20 points) You have an infinite supply of 4-cent coins and 5-cent coins. Note that you CAN create 9 cents (one 4-cent coin and one 5-cent coin) and 10 cents (two 5-cent coins) but you can't make 11 cents (proof: If you only use 4-cent coins then you can't make 11 since you can only make multiples of 4. If you only use 5-cent coins then you can't make 11 since you can only make multiples of 5. So you have to use at least one 4 and at least one 5. Hence you have 9. If you add 4 or 5 to that you exceed 11.)

   (a) Write a logical statement $P(n)$ for: "n coins can be created from 4-coins and 5-coins."

   (b) For $n = 4, 5, 6, \ldots$ see if n can be created from 4-coins and 5-coins. For each n that CAN be show how. For each n that CANNOT be prove why not. STOP when you are confident that you have an n_0 such that all $n \geq n_0$ CAN be.

   (c) For the value of n_0 that you determined in part (a), prove: *For all $n \geq n_0$ it is possible to create n cents with 4-coins and 5-coins.* (HINT: Use Strong Induction and you may need several base cases.)

   (d) **Challenge Problem.** Suppose we were to have x-coins and y-coins. Would such an n_0 still exist? Determine a necessary condition that x and y must satisfy in order for this to happen.

SOLUTION TO PROBLEM 4

(a) $P(n) : \exists x, y \in \mathsf{N}$ such that $n = 4x + 5y$.

(b)

4-YES 4=4.

5-YES 5=5

6-NO since can't just use one 4 or one 5, so need to use two coins, but that yields at least 8.

7-NO same reason as 7

8 YES 8 = 4+4

9 YES 9= 4+5

10 YES 10= 5+5

11 NO Can't use all 4's or all 5's since 11 is not div by 4 or 5. So have to use at least one of each. That gives you 9. If you use a 4 or a 5 you will exceed 9.

12 YES 12=4+4+4

13 YES 13=4+4+5

14 YES 14=4+5+5

15 YES 15=5+5+5

16 YES 16=4+4+4+4

I am confident that for all $n \geq 12$, n can be written as 4's and 5's.

(c)

Theorem: For all $n \geq 12$, n can be written as a sum of 4's and 5's.

Proof:

Base Case: We take $n = 12, 13, 14, 15, 16$ as base cases and they were proven above. KEY- we can assume that $n \geq 16$.

IH: Assume that for all m such that $12 \leq m \leq n - 1$ m can be written as the sum of 4's and 5's.

IS: Given $n \geq 16$ we want to show that n is the sum of 4's and 5's. KEY: Since $n \geq 16$ we have that $12 \leq n - 4 \leq n - 1$. Hence by the IH we know that $n - 4$ can be written as a sum of 4's and 5's. Say

$n - 4 = 4A + 5B$

so

$n = 4A + 4 + 5B = 4(A + 1) + 5B$.

So we have n as a sum of 4's and 5's.

(d) We are looking to see if, for any x and y, there exists an n_0 such that for all $n \geq n_0$, any n coins could be created from x-coins and y-coins. We claim that a necessary condition for this to happen is that $\gcd(x, y) = 1$.

Suppose that such an n_0 exist. Then there exist a_0, b_0 and a_1, b_1 such that:

$n_0 = a_0 \cdot x + b_0 \cdot y$

$n_0 + 1 = a_1 \cdot x + b_1 \cdot y$

Subtracting the two equations from each other gives us:

$1 = (a_1 - a_0) \cdot x + (b_1 - b_0) \cdot y$

Let $\gcd(x, y) = d$ for some d.

Then $d | (a_1 - a_0) \cdot x + (b_1 - b_0) \cdot y$ and so $d | 1$. Therefore, $d = 1$.

END OF SOLUTION TO PROBLEM 4

5. (20 points) Give a Context Free Grammar for the language

$$\{a^n b^{2n} : n \in \mathsf{N}\}$$

SOLUTION TO PROBLEM 5

$S \to aSbb$

$S \to e$

END OF SOLUTION TO PROBLEM 5

6. (20 points) Show that the following Context Free Grammar generates $L_{\#a>\#b} = \{w : \#_a(w) > \#_b(w)\}$.

ADJUSTMENT: LET G BE THE GRAMMAR AND $L(G)$ BE WHAT IT GENERATES. NEED ONLY PROVE THAT $L(G) \subseteq L_{\#a>\#b}$.

$S \to aT$

$S \to aS$

$S \to bSS$

$T \to aTb$

$T \to bTa$

$T \to TT$

$T \to e$

(HINT AND YOU CAN USE THIS: Note that T generates

$Leq = \{w : \#_a(w) = \#_b(w)\}$

(HINT: First prove that every derivation of a string of a's and b's of length n must have MORE a's than b's. This will be by induction. In doing this, pay attention to all possible FIRST rules in a derivation. Secondly, prove that every string of length n where there are more a's than b's can be generated. This is also by induction.)

SOLUTION TO PROBLEM 6

Call the grammar G.

Let $L_{\#a>\#b} = \{w : \#_a(w) > \#_b(w)\}$.

Theorem 1: $L(G) \subseteq L_{\#a>\#b}$

Proof: Let $S \Rightarrow_n \alpha$ mean that S generates α in n steps.

We show that for all n,if $S \Rightarrow_n \alpha$ (which can have S's and T's in it) then $\#_a(\alpha) > \#_b(\alpha)$. OH- THIS WILL NOT WORK- note that $S \to Sbb$ generates a string with two more b's then a's.

BETTER: Let $\#_{a+S}(\alpha) = \#_a(\alpha) + \#_S(\alpha)$.

We show that for all n,if $S \Rightarrow_n \alpha$ (which can have S's and T's in it) then $\#_{a+S}(\alpha) > \#_b(\alpha)$.

Base Case: $n = 1$: The only strings we can generate with one step are aT, aS, bSS all of which have more a's + S's then b's.

IH: If $S \Rightarrow_{n-1} \alpha'$ then $\#_{a+S}(\alpha') > \#_b(\alpha')$.

IS: Let $S \Rightarrow_n \alpha$ be a derivation of length n.

Assume $S \Rightarrow_n \alpha$. Look at the previous step:

$S \Rightarrow_{n-1} \alpha'$.

By the IH $\#_{a+S}(\alpha') > \#_b(\alpha')$

Since there is a derivation to get α there must be an S or a T in α' that has a rule applied to it.

Case 1: The rule is $S \to aT$. Replacing S by aT in α' keeps $\#_{a+S}$ and $\#_b$ the same, so have $\#_{a+S}(\alpha) > \#_b(\alpha)$.

Case 2: The rule is $S \to aS$. Replacing S by aS in α' ADDS one to $\#_{a+S}$ while keeping $\#_b$ the same. So have $\#_{a+S}(\alpha) > \#_b(\alpha)$.

Case 3: The rule is $S \to bSS$. Replacing S by bSS in α' ADDS TWO to $\#_{a+S}$ and ADDS ONE to $\#_b$. So have $\#_{a+S}(\alpha) > \#_b(\alpha)$.

Case 4: The rule is either $T \to aTb$ or $T \to bTa$. Replacing T by either aTb or bTa in α' ADDS ONE to $\#_{a+S}$ and ADDS ONE to $\#_b$. So have $\#_{a+S}(\alpha) > \#_b(\alpha)$.

Case 5: The rule is either $T \to TT$ or $T \to e$. Replacing T by either TT or e in α' does not change either $\#_{a+S}$ or $\#_b$. So have $\#_{a+S}(\alpha) > \#_b(\alpha)$.

End of Proof of Theorem 1.

Theorem 2: $L_{\#a>\#b} \subseteq L(G)$.

Proof: We show that every string $w \in L_{\#a>\#b}$ is in $L(G)$ by induction on $n = |w|$.

Base Case: $n = 1$. The only string of length 1 in $L_{\#a>\#b}$ is a which is in via $S \to aT$ and $T \to e$.

IH: All strings in $L_{\#a>\#b}$ of length $\le n-1$ are in $L(G)$.

IS: Let w be a string of length n in $L_{\#a>\#b}$.

Case 1: $w = aw'$ where $w' \in Leq$. Then use $S \to aT$ and then $T \Rightarrow w'$. The IH was not needed here.

Case 2: $w = aw'$ where $w' \in L_{\#a>\#b}$. Then use $S \to aS$ and then on the second S use $S \Rightarrow w'$ by the IH.

Case 3: $w = bw'$. w' must have at least 2 more a's than b's. Hence w' must be of the form $w''w'''$ where w'' and w''' are both in $L_{\#a>\#b}$. Use $S \to bSS$ and then the IH to get $S \Rightarrow w''$ and $S \Rightarrow w'''$.

END OF SOLUTION TO PROBLEM 6