Homework 9 Morally Due April 21
WARNING- THIS HW IS TWO PAGES LONG.

1. (0 points) Where and when is the final?

2. (25 points)

   (a) From a group of 7 men and 6 women, 5 people are to be selected to form a committee so that at least 3 men are on the committee. In how many ways can it be done?

   (b) If we didn't require that at least 3 men are on the committee (i.e. any number of men and women as long as they add up to 5), in how many ways could this be done? Try to use a different strategy from the one you used for part (a).

   (c) Show that, for any $n \in \mathsf{N}$, you have the following:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

   HINT: think of part (a) and (b) when there are n women, n men and the committee size is n.

3. (25 points) How many triplets of the form (x, y, z) are there from $\{1, 2, \ldots, n+1\}$ with $z > x$ and $z > y$? Let us denote this quantity as P_n. We will calculate P_n in 2 different ways:

   (a) Fix a value k for z, with $k \in \{1, 2, \ldots, n+1\}$. In how many ways can you choose the above x and y such that $x, y < k$? Use this quantity to derive a formula for P_n.

   (b) How many triples (x, y, z) are there from $\{1, 2, \ldots, n+1\}$ with $x = y < z$? What about $x < y < z$? Finally, what about $y < x < z$? Use these quantities to determine P_n.

   (c) Show, WITHOUT using induction, that:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

THERE ARE PROBLEMS ON THE NEXT PAGE

4. (25 points) Let $f(x) = x^2$. Note that I have not specified the domain or co-domain. For this problem domains and co-domains are subsets of the reals.

   (a) Give a domain and co-domain such that f is 1-1 AND onto (and prove it) OR show that no such exist.

   (b) Give a domain and co-domain such that f is 1-1 but NOT onto (and prove it) OR show that no such exist.

   (c) Give a domain and co-domain such that f is NOT 1-1 but is onto (and prove it) OR show that no such exist.

   (d) Give a domain and co-domain such that f is NOT 1-1 and NOT onto (and prove it) OR show that no such exist.

5. (25 points) Let n be a natural number. Let $f_n(x)$ be the function with domain $\{0, \ldots, n-1\}$ and co-domain $\{0, \ldots, n-1\}$ defined by $f_n(x) = x^3 \pmod{n}$.

   (a) Give a value $n \geq 5$ such that f_n is 1-1 AND onto OR show that no such exists.

   (b) Give a value $n \geq 5$ such that f_n is 1-1 but NOT onto OR show that no such exists.

   (c) Give a value $n \geq 5$ such that f_n is NOT 1-1 but is onto OR show that no such exists.

   (d) Give a value $n \geq 5$ such that f_n is NOT 1-1 and NOT onto. OR show that no such exists.