Homework 9 Morally Due April 21

WARNING- THIS HW IS TWO PAGES LONG.

1. (0 points) Where and when is the final?
2. (25 points)
   (a) From a group of 7 men and 6 women, 5 people are to be selected to form a committee so that at least 3 men are on the committee. In how many ways can it be done?
   (b) If we didn't require that at least 3 men are on the committee (i.e. any number of men and women as long as they add up to 5), in how many ways could this be done? Try to use a different strategy from the one you used for part (a).
   (c) Show that, for any $n \in \mathsf{N}$, you have the following:
   $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$
   HINT: think of part (a) and (b) when there are $n$ women, $n$ men and the committee size is $n$.

SOLUTION TO PROBLEM 2

(a) We will do divide and conquer. First, look at the case in which we select exactly 3 men and 2 women. This can be done in:

$$\binom{7}{3} \cdot \binom{6}{2} \text{ ways.}$$

Now consider the case in which we select 4 men and 1 woman. This can be done in:

$$\binom{7}{4} \cdot \binom{6}{1} \text{ ways.}$$

Finally, we consider the case in which we select 5 men and 0 women. This can be done in:

$$\binom{7}{5} \cdot \binom{6}{0} \text{ ways.}$$

So final answer:

$$\binom{7}{3} \cdot \binom{6}{2} + \binom{7}{4} \cdot \binom{6}{1} + \binom{7}{5} \cdot \binom{6}{0}.$$

(b) EASY BUT NOT VERY SMART WAY: in addition to the cases considered in (a), we also consider the cases in which we have 0 men and 5 women, 1 men and 4 women and 2 men and 3 women. This would give us in total:

$$\binom{7}{0} \cdot \binom{6}{5} + \binom{7}{1} \cdot \binom{6}{4} + \binom{7}{2} \cdot \binom{6}{3} + \binom{7}{3} \cdot \binom{6}{2} + \binom{7}{4} \cdot \binom{6}{1} + \binom{7}{5} \cdot \binom{6}{0}.$$

In other words, we would get:

$$\sum_{k=0}^{5} \binom{7}{k} \cdot \binom{6}{5-k.}$$

SMARTER WAY: Notice that now that we don't care how many men and women we pick as long as we pick 5 total, this problem just reduces to picking 5 people out of a group of $7+6 = 13$ people. In other words:

$$\binom{13}{5}.$$

(c) Notice that if we look at the two ways of solving (a) and (b), we would get that:

$$\sum_{k=0}^{5} \binom{7}{k} \cdot \binom{6}{5-k} = \binom{13}{5}.$$

If we consider $n$ men instead of 7 men, $n$ women instead of 5 and $n$ people in the committee instead of 5, we would get that:

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k} = \binom{2n}{n}.$$

Since we know that $\binom{n}{n-k} = \binom{n}{k}$, we get:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

3. (25 points) How many triplets of the form $(x, y, z)$ are there from $\{1, 2, \ldots, n+1\}$ with $z > x$ and $z > y$? Let us denote this quantity as $P_n$. We will calculate $P_n$ in 2 different ways:

   (a) Fix a value $k$ for $z$, with $k \in \{1, 2, \ldots, n+1\}$. In how many ways can you choose the above $x$ and $y$ such that $x, y < k$? Use this quantity to derive a formula for $P_n$.

   (b) How many triples $(x, y, z)$ are there from $\{1, 2, \ldots, n+1\}$ with $x = y < z$? What about $x < y < z$? Finally, what about $y < x < z$? Use these quantities to determine $P_n$.

   (c) Show, WITHOUT using induction, that:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

SOLUTION TO PROBLEM 3

(a) You have $k-1$ choices for $x$ and $k-1$ choices for $z$. Therefore there are $(k-1)^2$ total choices. We do that for every possible value of $k$ and get:

$$P_n = 1^2 + 2^2 + \ldots n^2 = \sum_{k=1}^{n} k^2.$$

(b) For the case when $x = y < z$, it is enough to choose 2 DISTINCT numbers from $\{1, 2, \ldots, n+1\}$. We set the smaller of the two to be the value of $x$ and $y$ and the larger of the two to be the value of $z$. So we get:

$$\binom{n+1}{2}.$$

For the case when $x < y < z$, we need to select 3 DISTINCT numbers from $\{1, 2, \ldots, n+1\}$. The smaller of the three will be the value for $x$, the middle one will be the value of $y$ and the larger one will be the value of $z$. We therefore get:

$$\binom{n+1}{3}.$$

Similarly, once we choose 3 DISTINCT numbers, we set the smaller of the three to be the value of $y$, the middle one to be the value of $x$ and the larger one to be the value of $z$. We again get:

$$\binom{n+1}{3}.$$

Now notice that, if we want a triplet $(x, y, z)$ such that $x, y < z$, then we will fall in one of the three cases mentioned above. So we get that:

$$P_n = \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}.$$

(c) From part (a) and (b), we get that:

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}.$$

So we only need to show that:

$$\binom{n+1}{2} + 2 \cdot \binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}.$$

We do this by using algebra:

$$\begin{aligned}
\binom{n+1}{2} + 2 \cdot \binom{n+1}{3} &= \frac{n(n+1)}{2} + 2 \cdot \frac{(n-1) \cdot n \cdot (n+1)}{6} \\
&= \frac{3n(n+1) + 2(n-1) \cdot n \cdot (n+1)}{6} \\
&= \frac{n(n+1)[3 + 2(n-1)]}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

**THERE ARE PROBLEMS ON THE NEXT PAGE**

4. (25 points) Let $f(x) = x^2$. Note that I have not specified the domain or co-domain. For this problem domains and co-domains are subsets of the reals.

   (a) Give a domain and co-domain such that $f$ is 1-1 AND onto (and prove it) OR show that no such exist.
   (b) Give a domain and co-domain such that $f$ is 1-1 but NOT onto (and prove it) OR show that no such exist.
   (c) Give a domain and co-domain such that $f$ is NOT 1-1 but is onto (and prove it) OR show that no such exist.
   (d) Give a domain and co-domain such that $f$ is NOT 1-1 and NOT onto (and prove it) OR show that no such exist.

SOLUTION TO PROBLEM 4

a) Domain and co-doman both $R^{>0}$ (the positive reals).

Proof:

1-1: if $x^2 = y^2$ then

$x^2 - y^2 = 0$

$(x - y)(x + y) = 0$

so either x=y or x=-y. Since we are operating over the positive reals has to be $x = y$. Hence $f$ is 1-1

Onto: Let $y \in R^{>0}$. Then $y$ has a square root, call it $x$. Clearly $f(x) = y$.

b) Domain is $R^{>0}$, co-domain is $R$ (all the reals).

Proof:

1-1 by same proof as part a.

NOT onto since no real maps to $-1$.

c) Domain is $R$ and co-domain is $R^{>0}$.

Proof:

NOT 1-1 since 2 and $-2$ both map to 4.

IS Onto since all positive reals have square roots.

d) Domain and co-domain are both $R$. Proof omitted but can be put together from whats above.

5. (25 points) Let $n$ be a natural number. Let $f_n(x)$ be the function with domain $\{0, \ldots, n-1\}$ and co-domain $\{0, \ldots, n-1\}$ defined by $f_n(x) = x^3 \pmod{n}$.

   (a) Give a value $n \geq 5$ such that $f_n$ is 1-1 AND onto OR show that no such exists.

   (b) Give a value $n \geq 5$ such that $f_n$ is 1-1 but NOT onto OR show that no such exists.

   (c) Give a value $n \geq 5$ such that $f_n$ is NOT 1-1 but is onto OR show that no such exists.

   (d) Give a value $n \geq 5$ such that $f_n$ is NOT 1-1 and NOT onto. OR show that no such exists.

   SOLUTION TO PROBLEM 5

   Before even beginning, look at the case of $n = 5$ since it's easy to compute and will have to satisfy one of these.

   Let $f = f_5$.

   $f_5(0) = 0$

   $f_5(1) = 1$

   $f_5(2) = 3$

   $f_5(3) = 2$

   $f_5(4) = 4 \times 4 \times 4 = -1 \times -1 \times -1 = -1 = 4$

   so this alread gives us 1-1 and onto!

   a) $n = 5$ as above.

   b) CANNOT HAPPEN. If a function from a finite set to itself is 1-1 then it is onto.

   c) CANNOT HAPPEN. If a function from a finite set to itself is onto then it is 1-1.

d) Want neither. All we need is NOT 1-1 and we get NOT onto.

Easiest way: we want a value of $n$ such that some $0 < m < n$, when cubed, gives 0 mod $n$.

$n = 8$.

$f(0) = 0$

$f(2) = 2^3 = 0$.

So $n = 8$ works.